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# The cyclotomic trace and curves on K-theory

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## Abstract

We give a functorial description of the topological cyclic homology of a ring  $A$  in terms of the relative algebraic K-theory of the truncated polynomial rings  $A_n = A[x]/x^n$ . This description involves the projection and transfer maps relating the relative K-theory spectra  $\tilde{K}(A_n)$  when  $n$  varies. From this point of view the cyclotomic trace corresponds to multiplication by the units  $1 + x + \cdots + x^{n-1}$  in  $\tilde{K}_1(\mathbb{Z}[x]/x^n)$ .

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## 1. Introduction

The cyclotomic trace introduced by Bökstedt et al. [7] is a natural map of spectra

$$\mathrm{trc}: K(A) \rightarrow \mathrm{TC}(A), \quad (1.1)$$

whose domain and target is respectively the algebraic K-theory spectrum and the topological cyclic homology spectrum associated to the unital ring  $A$ . The definition of  $\mathrm{TC}(A)$  is based on Bökstedt's topological Hochschild homology spectrum  $\mathrm{TH}(A)$ , [5,23]. Roughly speaking,  $\mathrm{TH}(A)$  is obtained by replacing the tensor products in the standard Hochschild complex by smash products of the Eilenberg–Mac Lane spectrum associated to  $A$ . The cyclic structure of the Hochschild complex provides  $\mathrm{TH}(A)$  with an action of the circle group  $\mathbb{T}$ , and by restriction with an action of the finite cyclic group  $C_r$  for each  $r \geq 1$ . Using that traces of  $r$ -fold matrix powers are fixed under cyclic permutation, the Bökstedt trace map may

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be lifted to maps

$$\mathrm{tr}^{C_r}: K(A) \rightarrow \mathrm{TH}(A)^{C_r}.$$

There are two types of structure maps

$$F_s, R_s: \mathrm{TH}(A)^{C_{rs}} \rightarrow \mathrm{TH}(A)^{C_r},$$

called, respectively, the Frobenius and the restriction maps. The Frobenius maps are the natural fixed-point inclusions, whereas the restriction maps result from the *cyclotomic structure* of  $\mathrm{TH}(A)$ , see [7,10,15]. The relations between these structure maps are formalized by introducing the category  $\mathbb{I}$  with objects the natural numbers  $1, 2, \dots$ , and morphisms generated by two types of arrows  $F_s, R_s: rs \rightarrow s$ , subject to the relations

$$R_1 = F_1 = \mathrm{id}, \quad R_r R_s = R_{rs}, \quad F_r F_s = F_{rs}, \quad R_r F_s = F_s R_r. \quad (1.2)$$

Any morphism can thus be uniquely written in the form  $F_s R_r$ . The correspondence  $r \mapsto \mathrm{TH}(A)^{C_r}$  then defines an  $\mathbb{I}$ -diagram and by definition,

$$\mathrm{TC}(A) = \mathrm{holim}_{\mathbb{I}} \mathrm{TH}(A)^{C_r}.$$

The maps  $\mathrm{tr}^{C_r}$  define a map from the constant diagram  $K(A)$  and assemble to give the cyclotomic trace in (1.1). This is a strong invariant of algebraic K-theory. In particular, Hesselholt and Madsen [15] have proved that the associated map of connective covers is a  $p$ -adic equivalence if  $A$  is a finite algebra over the Witt vectors of a perfect field of characteristic  $p > 0$ . This applies for example if  $A$  is a finite algebra over the  $p$ -adic integers. The methods of equivariant homotopy theory have then been used to calculate  $\mathrm{TC}(A)$  and thus  $K(A)$  in a number of cases, see e.g. [16,17,24]. However, there are also rings for which  $K(A)$  is better understood than  $\mathrm{TC}(A)$ . For example, the “fundamental theorem of algebraic K-theory” states that  $K(A[x, x^{-1}])$  decomposes into a wedge of  $K(A)$  and  $\Sigma K(A)$  when  $A$  is regular, whereas  $\mathrm{TC}(A[x, x^{-1}])$  at present is not well understood. We emphasize that the connection to Waldhausen’s algebraic K-theory of spaces via Dundas’ Theorem [8] makes  $\mathrm{TC}(A)$  an interesting functor also in cases where it does not agree with  $K(A)$ . The purpose of this paper is to show that in general the completion  $\mathrm{TC}(A)^\wedge$  has a natural description in terms of the algebraic K-theory spectra associated to the truncated polynomial rings  $A_n = A[x]/x^n$ . Let  $\tilde{K}(A_n)$  denote the homotopy fiber of the projection  $K(A_n) \rightarrow K(A)$ . The correspondence  $n \mapsto \tilde{K}(A_n)$  defines an  $\mathbb{I}$ -diagram in which the generating morphisms act via the spectrum maps

$$p_n, t_n: \tilde{K}(A_{mn}) \rightarrow \tilde{K}(A_m),$$

where  $p_n$  is induced by the projection  $A_{mn} \rightarrow A_m$  and  $t_n$  is the K-theoretical transfer associated to the homomorphism  $A_m \rightarrow A_{mn}$  taking  $x$  to  $x^n$ . The following is our main result.

**Theorem 1.3.** *For any unital ring  $A$  there is a natural equivalence of completed spectra*

$$\mathrm{TC}(A)^\wedge \simeq \mathrm{holim}_{\mathbb{I}} \Omega \tilde{K}(A_n)^\wedge.$$

The homotopy limit on the right hand side has an equivalent description

$$\operatorname{holim}_{\mathbb{I}} \Omega \tilde{K}(A_n) \simeq \Omega \tilde{K}^{\operatorname{top}}(A[[x]])^{h\{t_n\}}$$

as the homotopy fixed-point spectrum of the action by the transfer maps on the continuous K-theory

$$\tilde{K}^{\operatorname{top}}(A[[x]]) = \operatorname{holim}_{p_n} \tilde{K}(A_n).$$

This follows formally from the definition of the homotopy limits, see Section 2.3. Based on the K-theoretical tensor product pairings

$$\tilde{K}(\mathbb{Z}_n) \wedge K(A) \rightarrow \tilde{K}(A_n),$$

where  $\mathbb{Z}_n = \mathbb{Z}[x]/x^n$ , the cyclotomic trace has a simple description from the point of view of Theorem 1.3. We show that the classes in  $\tilde{K}_1(\mathbb{Z}_n)$  represented by the units  $u_n = 1 + x + \cdots + x^{n-1}$  assemble to give a stable homotopy class

$$u_{\infty}: \Sigma^{\infty}(S^1) \rightarrow \operatorname{holim}_{\mathbb{I}} \tilde{K}(\mathbb{Z}_n)$$

and multiplying by this class we get a stable map

$$u_{\infty}: \Sigma K(A) \rightarrow \operatorname{holim}_{\mathbb{I}} \tilde{K}(A_n).$$

**Addendum 1.4.** Under the equivalence in Theorem 1.3, the cyclotomic trace corresponds to  $\Omega(u_{\infty})$ .

We also remark that by the work of Dundas and McCarthy [9],  $\operatorname{TH}(A)$  is equivalent to Waldhausen's stable algebraic K-theory  $K^s(A)$ . It follows from [29, Section 6] that  $K^s(A)$  may be identified with the homotopy colimit

$$\operatorname{hocolim}_n \Omega^{n+1} \tilde{K}(A \rtimes A(S^n)),$$

where again  $\tilde{K}$  denotes relative K-theory. The initial term is  $\Omega \tilde{K}(A_2)$ , and projection followed by stabilization gives a map

$$\operatorname{holim}_{\mathbb{I}} \Omega \tilde{K}(A_n) \rightarrow \Omega \tilde{K}(A_2) \rightarrow K^s(A).$$

Under the equivalence in Theorem 1.3, this map corresponds to the canonical projection  $\operatorname{TC}(A) \rightarrow \operatorname{TH}(A)$ .

Our study of the truncated polynomial rings  $A_n$  in connection with the cyclotomic trace is motivated by the work of Bloch [3] and Hesselholt [14] on the curves

$$C(A) = \operatorname{holim}_{p_n} \Omega \tilde{K}(A_n)$$

on algebraic K-theory. Hesselholt proves [14, 3.1.10] that for  $\mathbb{Z}/p^j$ -algebras,  $C(A) \simeq \operatorname{TR}(A)$ , where the latter denotes the homotopy limit of the fixed point spectra  $\operatorname{TH}(A)^{C_r}$  under the restriction maps. Using this equivalence, he then calculates the  $p$ -typical curves  $C(A; p)$  in terms of the de Rham-Witt complex  $W\Omega_A^*$ , when  $A$  is a smooth algebra over a perfect field of characteristic  $p > 0$ . We hope that our explicit description of the cyclotomic trace in Addendum 1.4 will be useful also in this context.

Among other things this paper makes rigorous some of the constructions in the uncompleted preprint by the first author [2]. While working on this project we have benefited greatly from the hospitality of the SFB 343 at Bielefeld University.

## 2. Preliminaries on the cyclotomic trace

In this section we recall the version of the cyclotomic trace constructed by Dundas–McCarthy [10]. By a spectrum we understand a sequence of based spaces  $E_n$  for  $n \geq 0$ , together with a sequence of based structure maps  $S^1 \wedge E_n \rightarrow E_{n+1}$ . A map of spectra is supposed to commute strictly with the structure maps. It is an equivalence if it induces an equivalence on homotopy groups, the latter defined by  $\pi_i E = \operatorname{colim}_n \pi_{n+i}(E_n)$ . A symmetric spectrum  $E$  in the sense of [18] (in the setting of simplicial sets) and [20] is a spectrum in which each of the spaces  $E_n$  is equipped with an action of the symmetric group  $\Sigma_n$  such that the iterated structure maps  $S^m \wedge E_n \rightarrow E_{m+n}$  are  $\Sigma_m \times \Sigma_n$ -equivariant. The category of symmetric spectra has a symmetric monoidal smash product characterized by the property that a map of symmetric spectra  $E \wedge E' \rightarrow E''$  amounts to a compatible family of  $\Sigma_m \times \Sigma_n$ -equivariant maps  $E_m \wedge E_n \rightarrow E_{m+n}$ , see [20,22].

### 2.1. Algebraic K-theory

Let  $A$  be a (not necessarily commutative) ring with unit and let  $\mathcal{F}(A)$  be the full subcategory of the category of free left  $A$ -modules whose objects have the standard form  $A^n$ . We view  $\mathcal{F}(A)$  as an exact category in the usual way and define the algebraic K-theory spectrum  $K(A)$  by iterating Waldhausen's  $S_\bullet$ -construction, that is, the  $n$ th space  $K(A, n)$  is the topological realization of the multi-simplicial set of objects in the multi-simplicial exact category  $S_\bullet^n \mathcal{F}(A)$ . In particular, we define  $K(A, 0)$  to be the based set of objects in  $\mathcal{F}(A)$ . This definition is justified by [30, 1.4.1], which shows that the inclusion of  $K(A, n)$  as the zero simplices in the classifying space of the isomorphism subcategory  $iS_\bullet^n \mathcal{F}(A)$  is an equivalence. The symmetric group  $\Sigma_n$  acts on  $S_\bullet^n \mathcal{F}(A)$  in a natural way, giving  $K(A)$  the structure of a symmetric spectrum, see [11, 6.1]. This definition of algebraic K-theory has good multiplicative properties as we now recall. Given rings  $A$  and  $B$ , we have the bi-exact tensor product pairing

$$\mathcal{F}(A) \times \mathcal{F}(B) \rightarrow \mathcal{F}(A \otimes B), \quad (A^m, B^n) \mapsto (A \otimes B)^{mn}.$$

The effect on morphisms is dependent on a choice of linear ordering of the canonical basis of  $A^m \otimes B^n$ , that is, of an ordering of the product set

$$\{(i, j): 1 \leq i \leq m, 1 \leq j \leq n\}.$$

In this paper we shall always order such products using either the *lexicographical ordering*

$$(i_1, j_1) < (i_2, j_2) \Leftrightarrow i_1 < i_2 \quad \text{or} \quad i_1 = i_2 \text{ and } j_1 < j_2 \quad (2.1)$$

or the *reverse lexicographical ordering*,

$$(i_1, j_1) < (i_2, j_2) \Leftrightarrow j_1 < j_2 \quad \text{or} \quad j_1 = j_2 \text{ and } i_1 < i_2. \quad (2.2)$$

We shall always be explicit about which choice of ordering we use. It follows from the discussion in [11, 6.1] that with any of the two choices, the above exact pairing induces an exterior pairing of the associated symmetric algebraic K-theory spectra,

$$K(A) \wedge K(B) \rightarrow K(A \otimes B).$$

Giving a ring homomorphism  $A \otimes B \rightarrow C$ , we thus get an induced pairing of symmetric spectra  $K(A) \wedge K(B) \rightarrow K(C)$ .

## 2.2. Topological cyclic homology

We now recall the definition of the topological cyclic homology spectrum of the category  $\mathcal{F}(A)$ , referring the reader to [10,11] for more details. Consider more generally an additive category  $\mathcal{C}$ . Given objects  $c, d$  in  $\mathcal{C}$  and a finite based set  $X$ , let  $d \otimes \mathbb{Z}(X)$  be the coproduct of a family of copies of  $d$  indexed by  $X - \{*\}$ , and let  $\mathcal{C}(c, d \otimes \mathbb{Z}(X))$  be the abelian group of morphisms in  $\mathcal{C}$ . We extend this to simplicial  $X$  by topological realization of the simplicial abelian group obtained by applying this construction degree-wise. For each  $k \geq 0$ , let  $V_k[\mathcal{C}]$  be the  $(k+1)$ -fold multi-symmetric spectrum that to a  $(k+1)$ -tuple  $(n_0, \dots, n_k)$  associates the space

$$\bigvee_{c_0, \dots, c_k} \mathcal{C}(c_0, c_k \otimes \mathbb{Z}(S^{n_0})) \wedge \mathcal{C}(c_1, c_0 \otimes \mathbb{Z}(S^{n_1})) \wedge \dots \wedge \mathcal{C}(c_k, c_{k-1} \otimes \mathbb{Z}(S^{n_k})).$$

The wedge is over all  $(k+1)$ -tuples of objects in  $\mathcal{C}$ . Applying this construction degree-wise to the multi-simplicial additive category  $S_\bullet^n \mathcal{F}(A)$  we get a multi-simplicial object  $V_k[S_\bullet^n \mathcal{F}(A)]$  for each  $k$ . Let  $\mathcal{J}$  be the category whose objects are the finite sets  $n = \{1, \dots, n\}$  for  $n \geq 0$ , (0 is the empty set), and whose morphisms are the injective maps. Following [11] we define the topological Hochschild homology spectrum  $\mathrm{TH}(A)$  to be the symmetric spectrum whose  $n$ th space is the realization of the multi-simplicial cyclic space

$$[k] \mapsto \operatorname{hocolim}_{\mathcal{J}^{n(k+1)}} \operatorname{Map}(S^{\sqcup \vec{n}_0} \wedge \dots \wedge S^{\sqcup \vec{n}_k}, V_k[S_\bullet^n \mathcal{F}(A)](\sqcup \vec{n}_0, \dots, \sqcup \vec{n}_k))$$

with the usual Hochschild type structure maps. Here each  $\vec{n}_i$  denotes an object in  $\mathcal{J}^n$  and  $\sqcup : \mathcal{J}^n \rightarrow \mathcal{J}$  is the concatenation functor. The symmetric group  $\Sigma_n$  acts diagonally on  $\mathcal{J}^n$  and  $S_\bullet^n$ . The purpose of introducing the  $n$ -fold product category  $\mathcal{J}^n$  is to make the construction strictly multiplicative in the sense that a ring homomorphism  $A \otimes B \rightarrow C$  gives rise to a pairing of symmetric spectra

$$\mathrm{TH}(A) \wedge \mathrm{TH}(B) \rightarrow \mathrm{TH}(C)$$

as in the case of algebraic K-theory, see [11, 6.2]. The cyclic structure makes  $\mathrm{TH}(A)$  a spectrum with  $\mathbb{T}$ -action and by restriction a spectrum with  $C_r$ -action for each finite cyclic group  $C_r$ . As explained in Section 1, there are two types of structure maps

$$F_s, R_s : \mathrm{TH}(A)^{C_{rs}} \rightarrow \mathrm{TH}(A)^{C_r}.$$

Here the Frobenius maps  $F_r$  are the natural inclusions. The only things we shall need to know about the restriction maps  $R_r$  is that they (i) act trivially on the  $\mathbb{T}$ -fixed points of  $\mathrm{TH}(A)$ , (ii) are multiplicative with respect to the pairings introduced above, and (iii) together with the  $F_r$ -maps satisfy the relations (1.2).

Mapping an object in  $S_{\bullet}^n \mathcal{F}(A)$  to its identity morphism in the 0-skeleton of  $\mathrm{TH}(A, n)$ , we get a map of spectra

$$\mathrm{tr}: K(A) \rightarrow \mathrm{TH}(A).$$

The image is contained in the  $\mathbb{T}$ -fixed points of the right hand side, hence left invariant by the  $F_r$ - and  $R_r$ -maps. By definition, the cyclotomic trace is the induced map

$$\mathrm{trc}: K(A) \rightarrow \mathrm{holim}_{\mathbb{I}} \mathrm{TH}(A)^{C_r} = \mathrm{TC}(A).$$

It follows immediately from the definitions that the cyclotomic trace is strictly multiplicative in the sense that a ring homomorphism  $A \otimes B \rightarrow C$  gives rise to a strictly commutative diagram of pairings

$$\begin{array}{ccc} K(A) \wedge K(B) & \longrightarrow & K(C) \\ \downarrow \mathrm{trc} \wedge \mathrm{trc} & & \downarrow \mathrm{trc} \\ \mathrm{TC}(A) \wedge \mathrm{TC}(B) & \longrightarrow & \mathrm{TC}(C). \end{array}$$

In the analysis of  $\mathrm{TH}(A)$  it is often useful to introduce an additional spectrum coordinate and extend the definition of  $\mathrm{TH}(A)$  to give a symmetric bispectrum. In order to do this, let  $\mathrm{TH}(A, m, n)$  be the topological realization of the multi-simplicial cyclic space

$$[k] \mapsto \mathrm{hocolim}_{\mathcal{J}^{(m+n)(k+1)}} \mathrm{Map}(S^{\sqcup \vec{n}_0} \wedge \cdots \wedge S^{\sqcup \vec{n}_k}, V_k[S_{\bullet}^m \mathcal{F}(A)](\sqcup \vec{n}_0, \dots, \sqcup \vec{n}_k) \wedge S^n),$$

where each  $\vec{n}_i$  denotes an object of  $\mathcal{J}^{m+n}$ . It follows from [10, 2.1.3] that the adjoint spectrum maps are  $C_r$ -equivariant equivalences in positive bidegrees. Let  $\mathrm{TH}'(A)$  and  $\mathrm{TH}''(A)$  be the two spectra obtained by restricting to bidegrees  $(n, 0)$  and  $(0, n)$  respectively. Thus  $\mathrm{TH}'(A)$  is now what was denoted  $\mathrm{TH}(A)$  earlier. It follows by a standard argument that these spectra are related by a chain of  $\mathbb{T}$ -equivariant spectrum maps

$$\mathrm{TH}'(A) \simeq \mathrm{TH}''(A),$$

each of which induces an equivalence on  $C_r$  fixed-points for all  $r$ . Passing to the homotopy limit over the associated  $\mathbb{I}$ -diagrams we thus get a chain of equivalences

$$\mathrm{TC}'(A) \simeq \mathrm{TC}''(A). \quad (2.3)$$

Furthermore, the multiplicative properties of  $\mathrm{TH}'(A)$  considered above extends to the symmetric bispectrum  $\mathrm{TH}(A)$  and the maps in the above chain of equivalences are multiplicative.

### 2.3. Homotopy limits of $\mathbb{I}$ -diagrams

Following Goodwillie [12], we now make some general remarks on homotopy limits of  $\mathbb{I}$ -diagrams. With the definition of Bousfield and Kan [4], the homotopy limit of a diagram of spaces may be constructed in two steps: First form the cosimplicial replacement of the diagram, then take the total space of this cosimplicial space. In particular, if  $M$  is a monoid acting from the left on a space  $X$ , then one defines the homotopy fixed-point space  $X^{hM}$  to be the homotopy limit of the diagram obtained by viewing  $X$  as a

functor from the category with one object in the usual way. There is a natural homeomorphism (induced from an isomorphism of the underlying cosimplicial spaces)

$$X^{hM} \cong \text{Map}_M(EM, X),$$

where  $EM = B(M, M, *)$  is the realization of the one sided bar construction and  $\text{Map}_M(-, -)$  denotes the space of  $M$ -equivariant maps. Let  $\tilde{M}$  be the (right) translation category associated with  $M$ : The objects are the elements of  $M$ , and a morphism  $a \leftarrow c$  is an element  $b$  in  $M$  such that  $ab = c$ . The classifying space of  $\tilde{M}$  is homeomorphic to  $EM$ , and the following lemma can be checked on the level of cosimplicial spaces.

**Lemma 2.4.** *Suppose that  $Y$  is an  $\tilde{M}$ -diagram in the category of spaces. Then the monoid  $M$  acts on the product  $\prod_{m \in M} Y_m$  by*

$$a \cdot (y_m)_{m \in M} = (a_*(y_{ma}))_{m \in M},$$

and

$$\text{holim}_{\tilde{M}} Y \cong \left( \prod_{m \in M} Y_m \right)^{hM}.$$

Let now  $X$  be an  $\mathbb{I}$ -diagram in the category of spaces, and let  $N$  be the multiplicative monoid of (positive) natural numbers. The actions by the maps  $F_r$  and  $R_r$  give two commuting actions of  $N$  on the product  $\prod_{n=1}^{\infty} X_n$  and we denote the corresponding homotopy fixed-points by  $(-)^{h\{R_r\}}$  and  $(-)^{h\{F_r\}}$ . Since every morphism in  $\mathbb{I}$  can be written uniquely in the form  $F_r R_s$ , one has in analogy with Lemma 2.4 a homeomorphism

$$\text{holim}_{\mathbb{I}} X \cong \left( \prod_{n=1}^{\infty} X_n \right)^{h(\{F_r\} \times \{R_r\})}, \quad (2.5)$$

again induced from an isomorphism of cosimplicial sets. Forgetting the action of the  $R_r$ -maps, say,  $X$  restricts to an  $\tilde{N}$  functor. Using Lemma 2.4 and (2.5), we get that

$$\text{holim}_{\mathbb{I}} X \cong \left( \left( \prod_{n=1}^{\infty} X_n \right)^{h\{F_r\}} \right)^{h\{R_r\}} \cong \left( \text{holim}_{F_r} X_n \right)^{h\{R_r\}}. \quad (2.6)$$

We extend the above remarks to spectra by evaluating the homotopy limits in each spectrum degree. In particular, letting

$$\text{TF}(A) = \text{holim}_{F_r} \text{TH}(A)^{C_r}, \quad \text{TR}(A) = \text{holim}_{R_r} \text{TH}(A)^{C_r},$$

we get that

$$\text{TC}(A) \cong \text{TF}(A)^{h\{R_r\}} \cong \text{TR}(A)^{h\{F_r\}}.$$

In the following we shall only consider homotopy limits of positive  $\Omega$ -spectra, i.e., spectra in which the adjoint structure maps are equivalences in positive degrees. In this case the homotopy limit of a natural equivalence of diagrams is again an equivalence.

## 2.4. Completions

We shall often consider the completion  $E^\wedge$  of a spectrum  $E$ . Choosing functorial models for the Moore spectra  $M(\mathbb{Z}/n)$ , this may be defined as the homotopy inverse limit

$$E^\wedge = \operatorname{holim}_{\tilde{N}} E \wedge M(\mathbb{Z}/n),$$

where  $\tilde{N}$  is as above. This is equivalent to the product over all primes  $p$  of the  $p$ -completions  $E_p^\wedge$ . With this definition it is clear that homotopy limits commute with completion. The following lemma will be needed later.

**Lemma 2.7.** *Suppose given an  $\tilde{N}$ -diagram in the category of spectra that is constantly equal to  $E$  on objects and has the property that the morphism  $rs \rightarrow r$  induces multiplication by  $s$  on the homotopy groups of  $E$ . Then the completion of  $\operatorname{holim}_{\tilde{N}} E$  is contractible.*

## 3. Diagrams of truncated polynomial rings

Given a unital ring  $A$ , we consider the associated truncated polynomial rings  $A_n = A[x]/x^n$ . The natural projections  $p_n: A_{mn} \rightarrow A_m$  give rise to exact functors

$$p_n: \mathcal{F}(A_{mn}) \rightarrow \mathcal{F}(A_m), \quad A_{mn}^r \mapsto A_m^r$$

in which the effect on morphisms is determined by the canonical isomorphisms of  $A_m$ -modules  $A_m \otimes_{A_{mn}} A_{mn}^r \cong A_m^r$ . These functors are transitive in the sense that  $p_m p_n = p_{mn}$ . Let  $i_n: A_m \rightarrow A_{mn}$  be the ring homomorphism determined by  $x \mapsto x^n$ . Since  $A_{mn}$  is free as an  $A_m$ -module via  $i_n$ , there are associated restriction (or transfer) functors

$$t_n: \mathcal{F}(A_{mn}) \rightarrow \mathcal{F}(A_m), \quad A_{mn}^r \mapsto A_m^{nr}. \quad (3.1)$$

The effect on morphisms depends on a choice of  $A_m$ -linear isomorphisms  $A_{mn}^r \cong A_m^{nr}$  for  $r \geq 0$ . The elements  $\{1, x, \dots, x^{n-1}\}$  form an ordered basis for  $A_{mn}$  as an  $A_m$ -module and determine an isomorphism of  $A_m$ -modules of the form  $\phi: A_{mn} \rightarrow A_m^n$ . There are induced isomorphisms

$$\bigoplus_{i=1}^r \phi: \bigoplus_{i=1}^r A_{mn} \rightarrow \bigoplus_{i=1}^r \bigoplus_{j=1}^n A_m, \quad (3.2)$$

and we must identify the target with  $A_m^{nr}$ . We do this by ordering the indexing set

$$\{(i, j) : 1 \leq i \leq r, 1 \leq j \leq n\}$$

using the reverse lexicographical ordering as in (2.2). With this choice of ordering we again have strict transitivity,  $t_m t_n = t_{mn}$ . (Using the lexicographical ordering this would only hold up to natural isomor-



phism). Furthermore, these functors are related by strictly commutative diagrams of the form

$$\begin{array}{ccc} \mathcal{F}(A_{lmn}) & \xrightarrow{t_n} & \mathcal{F}(A_{lm}) \\ \downarrow p_m & & \downarrow p_m \\ \mathcal{F}(A_{ln}) & \xrightarrow{t_n} & \mathcal{F}(A_l). \end{array}$$

Recall the category  $\mathbb{I}$  from Section 1. The above discussion may be summarized as follows.

**Proposition 3.3.** *Letting the morphisms  $R_n$  and  $F_n$  act via  $p_n$  and  $t_n$  respectively, the correspondence  $n \mapsto \mathcal{F}(A_n)$  defines an  $\mathbb{I}$ -diagram of exact categories.*

It follows that any functor taking exact categories as input gives rise to an  $\mathbb{I}$ -diagram by evaluating it on the categories  $\mathcal{F}(A_n)$ . In particular, we get such diagrams by applying algebraic K-theory and topological cyclic homology, and the cyclotomic trace defines a natural transformation of  $\mathbb{I}$ -diagrams

$$\mathrm{trc}: K(A_n) \rightarrow \mathrm{TC}(A_n).$$

Let  $\tilde{K}(A_n)$  be the homotopy fiber of the natural projection  $K(A_n) \rightarrow K(A)$ . We may view the latter as a natural transformation of  $\mathbb{I}$ -diagrams in which the actions of  $p_n$  and  $t_n$  on  $K(A)$  are induced respectively by the identity and the  $n$ -fold direct sum functor on  $\mathcal{F}(A)$ . It follows that the correspondence  $n \mapsto \tilde{K}(A_n)$  inherits the structure of an  $\mathbb{I}$ -diagram. Similarly, we let  $\tilde{\mathrm{TC}}(A_n)$  be the homotopy fiber of the corresponding map  $\mathrm{TC}(A_n) \rightarrow \mathrm{TC}(A)$  and consider the induced map of  $\mathbb{I}$ -diagrams

$$\mathrm{trc}: \tilde{K}(A_n) \rightarrow \tilde{\mathrm{TC}}(A_n). \quad (3.4)$$

By McCarthy's Theorem [22], the completion of this is an equivalence for each  $n$ . Passing to homotopy limits we thus get the following.

**Proposition 3.5.** *The cyclotomic trace induces an equivalence*

$$\mathrm{trc}: \mathrm{holim}_{\mathbb{I}} \tilde{K}(A_n)^{\wedge} \xrightarrow{\sim} \mathrm{holim}_{\mathbb{I}} \tilde{\mathrm{TC}}(A_n)^{\wedge}$$

*of completed spectra.*

Consider now the natural ring homomorphisms

$$\mathbb{Z}_n \otimes A \rightarrow A_n$$

(where as usual  $\mathbb{Z}_n = \mathbb{Z}[x]/x^n$ ), and the corresponding bi-exact pairings

$$\mathcal{F}(\mathbb{Z}_n) \times \mathcal{F}(A) \rightarrow \mathcal{F}(A_n). \quad (3.6)$$

Here we use the lexicographical ordering as in (2.1). This choice makes the pairings compatible with the  $p_n$  and  $t_n$  functors in the sense of the following lemma.

**Lemma 3.7.** *The bi-exact pairings in (3.6) define a pairing of  $\mathbb{I}$ -diagrams of exact categories in which we view  $\mathcal{F}(A)$  as the constant diagram.*

Passing to algebraic K-theory, the above exact functors induce a map of  $\mathbb{I}$ -diagrams

$$K(\mathbb{Z}_n) \wedge K(A) \rightarrow K(A_n).$$

As usual, such a pairing of symmetric spectra amounts to a compatible family of  $\Sigma_m \times \Sigma_n$ -equivariant maps. This in turn induces a map of  $\mathbb{I}$ -diagrams of the relative theories

$$\tilde{K}(\mathbb{Z}_n) \wedge K(A) \rightarrow \tilde{K}(A_n). \quad (3.8)$$

We shall consider the units  $u_n = 1 + x + \cdots + x^{n-1}$  in  $\mathbb{Z}_n$  and the corresponding classes  $u_n$  in  $\tilde{K}_1(\mathbb{Z}_n)$ . It is elementary to check that these classes are preserved under the actions of  $p_n$  and  $t_n$ , hence define an element  $u_\infty$  in  $\lim_{\mathbb{I}} \tilde{K}_1(\mathbb{Z}_n)$ .

**Lemma 3.9.** *The classes  $u_n$  lift to a stable map*

$$u_\infty: \Sigma^\infty(S^1) \rightarrow \operatorname{holim}_{\mathbb{I}} \tilde{K}(\mathbb{Z}_n).$$

**Proof.** We shall consider the exact category  $\operatorname{End}(\mathbb{Z})$  in which an object  $(P, \alpha)$  is a free  $\mathbb{Z}$ -module  $P$  together with an endomorphism  $\alpha: P \rightarrow P$ . A morphism between two such objects  $(P, \alpha)$  and  $(Q, \beta)$  is a homomorphism  $\gamma: P \rightarrow Q$  such that  $\beta\gamma = \gamma\alpha$ . A sequence in  $\operatorname{End}(\mathbb{Z})$  is exact if the underlying sequence of free  $\mathbb{Z}$ -modules is. Let  $\tilde{K}(\operatorname{End}(\mathbb{Z}))$  be the homotopy fiber of the map  $K(\operatorname{End}(\mathbb{Z})) \rightarrow K(\mathbb{Z})$  induced by the forgetful functor  $(P, \alpha) \mapsto P$ . By Grayson's theorem [13,1], there is an equivalence of spectra

$$\tilde{K}(\operatorname{End}(\mathbb{Z})) \simeq \Omega\tilde{K}(R),$$

where  $R$  is the augmented ring obtained by localizing  $\mathbb{Z}[x]$  at the multiplicative subset  $1 + x\mathbb{Z}[x]$ . Furthermore, the Frobenius morphisms act on  $\operatorname{End}(\mathbb{Z})$  by  $F_n(P, \alpha) = (P, \alpha^n)$  and this action is compatible with the action of the transfer maps  $t_n$  on  $\Omega\tilde{K}(R)$ . More precisely, it follows from the proof of [28, 4.7] that the above equivalence may be realized by a chain of spectrum maps each of which commutes strictly with these operations. On the level of homotopy groups we thus get a homomorphism

$$\pi_0 \tilde{K}(\operatorname{End}(\mathbb{Z}))^{h\{F_n\}} \xrightarrow{\sim} \pi_1 \tilde{K}(R)^{h\{t_n\}} \rightarrow \pi_1 \operatorname{holim}_{\mathbb{I}} \tilde{K}(\mathbb{Z}_n),$$

where, using (2.6), the last map is induced by the projections  $R \rightarrow \mathbb{Z}_n$ . The two objects  $(\mathbb{Z}, 0)$  and  $(\mathbb{Z}, \operatorname{id})$  are fixed by the  $\{F_n\}$ -action, hence the difference  $(\mathbb{Z}, 0) - (\mathbb{Z}, \operatorname{id})$  determines a class in  $\pi_0 \tilde{K}(\operatorname{End}(\mathbb{Z}))^{h\{F_n\}}$ . We let  $u_\infty$  be the image of this class under the above homomorphism. It follows from [13,3] that this is indeed a lift of the classes  $u_n$  as required.  $\square$

Passing to homotopy limits, the pairings in (3.8) induce a pairing

$$\operatorname{holim}_{\mathbb{I}} \tilde{K}(\mathbb{Z}_n) \wedge K(A) \rightarrow \operatorname{holim}_{\mathbb{I}} \tilde{K}(A_n)$$

and multiplying by  $u_\infty$  we get a stable map

$$u_\infty: \Sigma K(A) \rightarrow \operatorname{holim}_{\mathbb{I}} \tilde{K}(A_n). \quad (3.10)$$

Let  $u_\infty^{\operatorname{TC}}$  be the stable homotopy class defined by the composition

$$u_\infty^{\operatorname{TC}}: \Sigma^\infty(S^1) \xrightarrow{u_\infty} \operatorname{holim}_{\mathbb{I}} \tilde{K}(\mathbb{Z}_n) \rightarrow \operatorname{holim}_{\mathbb{I}} \tilde{\operatorname{TC}}(\mathbb{Z}_n).$$

Using the TC-pairings induced by the bi-exact functors in (3.6), we multiply by this stable class to get a TC-analogue of (3.10),

$$u_{\infty}^{\text{TC}}: \Sigma\text{TC}(A) \rightarrow \text{holim}_{\mathbb{I}} \tilde{\text{TC}}(A_n).$$

By the multiplicativity of the cyclotomic trace there results a commutative diagram

$$\begin{array}{ccc} \Sigma K(A) & \xrightarrow{\Sigma \text{trc}} & \Sigma \text{TC}(A) \\ \downarrow u_{\infty} & & \downarrow u_{\infty}^{\text{TC}} \\ \text{holim}_{\mathbb{I}} \tilde{K}(A_n) & \xrightarrow{\text{trc}} & \text{holim}_{\mathbb{I}} \tilde{\text{TC}}(A_n) \end{array} \quad (3.11)$$

in the stable homotopy category. We shall prove the following proposition in Section 6.

**Proposition 3.12.** *Multiplication by  $u_{\infty}^{\text{TC}}$  induces an equivalence*

$$u_{\infty}^{\text{TC}}: \Sigma\text{TC}(A)^{\wedge} \xrightarrow{\sim} \text{holim}_{\mathbb{I}} \tilde{\text{TC}}(A_n)^{\wedge}$$

*after completion.*

**Proof of Theorem 1.3.** By Propositions 3.5 and 3.12 we have the equivalences

$$\Sigma\text{TC}(A)^{\wedge} \xrightarrow{\sim} \text{holim}_{\mathbb{I}} \tilde{\text{TC}}(A_n)^{\wedge} \xleftarrow{\sim} \text{holim}_{\mathbb{I}} \tilde{K}(A_n)^{\wedge}.$$

The result then follows by applying  $\Omega$  and using that the latter commutes with homotopy limits.  $\square$

The description of the cyclotomic trace in Addendum 1.4 follows immediately from Diagram (3.11).

#### 4. Restriction maps in topological Hochschild homology

In this section we adapt the general theory of restriction maps in topological cyclic homology from [27] to the case of the restriction maps  $t_n$  introduced in Section 3. In order to apply the theory from [27], we shall in this section let  $\text{TH}(A)$  denote the model of the topological Hochschild homology spectrum that was denoted  $\text{TH}''(A)$  in Section 2.2, that is,  $\text{TH}(A)$  is the symmetric spectrum whose  $n$ th space is the realization of the cyclic space

$$[k] \mapsto \text{hocolim}_{\mathcal{J}^{n(k+1)}} \text{Map}(S^{\sqcup \vec{n}_0} \wedge \cdots \wedge S^{\sqcup \vec{n}_k}, V_k[F(A)](\sqcup \vec{n}_0, \dots, \sqcup \vec{n}_k) \wedge S^n).$$

As in [27] it will be important to extend this to a genuine  $\mathbb{T}$ -spectrum indexed on a complete set of real  $\mathbb{T}$ -representations. Let  $\mathcal{V}$  be the set of finite dimensional real  $\mathbb{T}$ -representations of the form

$$V = \mathbb{R}^{n_0} \oplus \mathbb{C}(1)^{n_1} \oplus \mathbb{C}(2)^{n_2} \dots,$$

where  $n_k$  is non-zero for only finitely many  $k$ . Here  $\mathbb{C}(k)$  denotes the 2-dimensional real representation obtained by letting an element  $z \in \mathbb{T}$  act on  $\mathbb{C}$  via multiplication by  $z^k$ . Let  $S^V$  be the one-point

compactification of  $V$ . Given  $n \geq 0$  and  $V$  as above, we define  $\mathrm{TH}(A, n, V)$  to be the realization of the cyclic space

$$[k] \mapsto \mathrm{hocolim}_{\mathcal{J}^{n(k+1)}} \mathrm{Map}(S^{\sqcup \vec{n}_0} \wedge \cdots \wedge S^{\sqcup \vec{n}_k}, V_k[\mathcal{F}(A)](\sqcup \vec{n}_0, \dots, \sqcup \vec{n}_k) \wedge S^n \wedge S^V).$$

Since by definition this is the realization of a cyclic  $\mathbb{T}$ -space, it has a natural  $\mathbb{T} \times \mathbb{T}$ -action in which one factor acts on  $S^V$  and the action of the other factor is induced by the cyclic structure. We view  $\mathrm{TH}(A, n, V)$  as a  $\mathbb{T}$ -space with the diagonal action. There are  $\mathbb{T}$ -equivariant structure maps

$$S^V \wedge \mathrm{TH}(A, n, W) \rightarrow \mathrm{TH}(A, n, V \oplus W)$$

that commute with the structure maps in the  $n$ -direction. In this way we may view  $\mathrm{TH}(A)$  as a symmetric spectrum in the category of  $\mathbb{T}$ -spectra.

#### 4.1. Topological Hochschild homology of reduced monoid rings

Let  $\Pi$  be a based monoid by which we mean a discrete monoid equipped with a base point such that the multiplication factors over the smash product. The main example for us will be the based monoid  $\Pi_n$  given by the set of monomials  $\{0, 1, x, \dots, x^{n-1}\}$  in  $\mathbb{Z}[x]/x^n$ . Given a ring  $A$ , we may identify  $A_n$  with the reduced monoid ring  $A(\Pi_n)/A\{0\}$ . As in [27] we shall consider the symmetric ring spectrum  $\mathbb{S}(\Pi)$  whose  $n$ th space is  $S^n \wedge \Pi$ . We view  $\mathbb{S}(\Pi)$  as the reduced monoid ring with coefficients in the sphere spectrum. The  $a \times b$  matrix spectrum  $M_{a,b}(\mathbb{S}(\Pi))$  is the symmetric spectrum

$$M_{a,b}(\mathbb{S}(\Pi))(n) = \mathrm{Map}(\mathbf{b}_+, \mathbf{a}_+ \wedge S^n \wedge \Pi),$$

where  $\mathbf{a}_+$  denotes the set  $\{1, \dots, a\}$  equipped with a disjoint base point and similarly for  $\mathbf{b}_+$ . Let  $V_k[\mathcal{F}(\mathbb{S}(\Pi))]$  be the  $(k+1)$ -fold multi-symmetric spectrum that to a  $(k+1)$ -tuple  $(n_0, \dots, n_k)$  associates the space

$$\bigvee_{a_0, \dots, a_k} M_{a_k, a_0}(\mathbb{S}(\Pi))(n_0) \wedge \cdots \wedge M_{a_{k-1}, a_k}(\mathbb{S}(\Pi))(n_k).$$

The wedge product is over all  $(k+1)$ -tuples of natural numbers. We define  $\mathrm{TH}(\mathbb{S}(\Pi))$  to be the symmetric spectrum of  $\mathbb{T}$ -spectra obtained by replacing  $V_k[\mathcal{F}(A)]$  by  $V_k[\mathcal{F}(\mathbb{S}(\Pi))]$  in the definition of  $\mathrm{TH}(A)$ . Using the linearization map

$$\mathrm{TH}(\mathbb{S}(\Pi)) \rightarrow \mathrm{TH}(\mathbb{Z}(\Pi)),$$

we get a pairing of symmetric spectra

$$\mathrm{TH}(\mathbb{S}(\Pi)) \wedge \mathrm{TH}(A) \rightarrow \mathrm{TH}(A(\Pi)).$$

This is a non-equivariant equivalence. In order to get an equivariant equivalence, we define the  $\mathbb{T}$ -equivariant smash product  $\mathrm{TH}(\mathbb{S}(\Pi)) \wedge_{\mathcal{V}} \mathrm{TH}(A)$  to be the symmetric  $\Omega$  bispectrum whose  $(m, n)$ th space is given by

$$\mathrm{hocolim}_{V, W} \Omega^{V \oplus W}(\mathrm{TH}(\mathbb{S}(\Pi), m, V) \wedge \mathrm{TH}(A, n, W)),$$

where the homotopy colimit is over all pairs of  $\mathbb{T}$ -representations of the standard form considered above, see [27] for details. Let  $\bar{\mathrm{TH}}(A(\Pi))$  be the symmetric spectrum

$$\mathrm{TH}(A(\Pi), n) = \operatorname{hocolim}_{V, W} \Omega^{V \oplus W}(\bar{\mathrm{TH}}(A, n, V \oplus W))$$

and consider the chain of maps of symmetric bispectra

$$\mathrm{TH}(\mathbb{S}(\Pi)) \wedge_{\mathcal{V}} \mathrm{TH}(A) \rightarrow \bar{\mathrm{TH}}(A(\Pi)) \leftarrow \mathrm{TH}(A(\Pi)). \quad (4.1)$$

Here we view  $\mathrm{TH}(A(\Pi))$  and  $\bar{\mathrm{TH}}(A(\Pi))$  as bispectra in the usual way by evaluating them on  $m + n$  in bidegree  $(m, n)$ .

**Proposition 4.2** (Schlichtkrull [27]). *The maps in (4.1) induce a chain of equivalences of fixed-point spectra*

$$(\mathrm{TH}(\mathbb{S}(\Pi)) \wedge_{\mathcal{V}} \mathrm{TH}(A))^{C_r} \simeq \mathrm{TH}(A(\Pi))^{C_r}$$

for each  $r \geq 1$ .  $\square$

We now specialize to the case of the based monoids  $\Pi_n$ . Mimicking the linear case, the correspondence  $n \mapsto \mathrm{TH}(\mathbb{S}(\Pi_n))$  defines an  $\mathbb{I}$ -diagram of  $\mathbb{T}$ -equivariant spectra. Let  $\tilde{\mathrm{TH}}(\mathbb{S}(\Pi_n))$  be the homotopy fiber of the projection  $\mathrm{TH}(\mathbb{S}(\Pi_n)) \rightarrow \mathrm{TH}(\mathbb{S})$ .

**Proposition 4.3.** *The chain of equivalences in Proposition 4.2 gives rise to a chain of equivalences of  $\mathbb{I}$ -diagrams*

$$(\tilde{\mathrm{TH}}(\mathbb{S}(\Pi_n)) \wedge_{\mathcal{V}} \mathrm{TH}(A))^{C_r} \simeq \tilde{\mathrm{TH}}(A_n)^{C_r}$$

for each  $r \geq 1$ .

**Proof.** It is clear from the definitions that the equivalences in Proposition 4.2 assemble to give a level-wise equivalence of  $\mathbb{I}$ -diagrams

$$(\mathrm{TH}(\mathbb{S}(\Pi_n)) \wedge_{\mathcal{V}} \mathrm{TH}(A))^{C_r} \simeq \mathrm{TH}(A_n)^{C_r}.$$

In particular, for  $n = 1$ , this gives a chain of equivalences relating the fixed-points of  $\mathrm{TH}(\mathbb{S}) \wedge_{\mathcal{V}} \mathrm{TH}(A)$  and  $\mathrm{TH}(A)$ . As in the linear case we may view the projections  $\mathrm{TH}(\mathbb{S}(\Pi_n)) \rightarrow \mathrm{TH}(\mathbb{S})$  as defining a map of  $\mathbb{I}$ -diagrams of  $\mathbb{T}$ -spectra. Forming the equivariant smash product with  $\mathrm{TH}(A)$  and evaluating the  $C_r$  fixed-points we get a fibration sequence up to homotopy and the result follows.  $\square$

In order to analyze the  $\mathbb{I}$ -diagram  $n \mapsto \tilde{\mathrm{TH}}(\mathbb{S}(\Pi_n))$  we proceed as in [27] by using the Barratt–Eccles construction to obtain a combinatorial model.

#### 4.2. The equivariant Barratt–Eccles suspension spectrum

Let  $E_{\bullet} \Sigma_n$  be the cyclic set

$$E_{\bullet} \Sigma_n: [k] \mapsto \operatorname{Map}([k], \Sigma_n) = \Sigma_n^{k+1}$$

with the usual simplicial structure and cyclic operators

$$t_k(\sigma_0, \dots, \sigma_k) = (\sigma_k, \sigma_0, \dots, \sigma_{k-1}).$$

Given a based space  $X$ , the Barratt–Eccles construction  $\mathbb{E}_{\bullet}^{\infty}(X)$  is the cyclic space

$$\mathbb{E}_{\bullet}^{\infty}(X) = \left( \prod_{n=0}^{\infty} E_{\bullet, \Sigma_n} \times X^n \right) / (\alpha^*(\mathbf{e}), \mathbf{x}) \sim (\mathbf{e}, \alpha_*(\mathbf{x})),$$

where  $\alpha$  runs through the morphisms in the category  $\mathcal{J}$  introduced in Section 2.2. Here we view  $\mathbf{n} \mapsto E_{\bullet, \Sigma_n}$  and  $\mathbf{n} \mapsto X^n$  as defining, respectively, a contravariant and a covariant functor, see e.g. [27]. Thus  $\mathbb{E}_{\bullet}^{\infty}(X)$  is by definition a coend in the sense of [19, IX.6]. We write  $\mathbb{E}^{\infty}(X)$  for the topological realization with the  $\mathbb{T}$ -action induced by the cyclic action. If  $X$  is itself a  $\mathbb{T}$ -space, then  $\mathbb{E}^{\infty}(X)$  is the realization of a cyclic  $\mathbb{T}$ -space and therefore has a natural  $\mathbb{T} \times \mathbb{T}$ -action. In this case we view it as a  $\mathbb{T}$ -space with the diagonal action. As in [27] we consider the symmetric spectrum with  $\mathbb{T}$ -action  $\mathbb{E}\Sigma_{\mathbb{T}}^{\infty}(X)$  whose  $n$ th space is given by

$$\mathbb{E}\Sigma_{\mathbb{T}}^{\infty}(X)(n) = \mathbb{E}^{\infty}(X \wedge S^n).$$

This has an obvious extension to a symmetric spectrum of  $\mathbb{T}$ -spectra with  $(n, V)$ th space  $\mathbb{E}^{\infty}(X \wedge S^n \wedge S^V)$ . We use the notation  $\bar{\Sigma}_{\mathbb{T}}^{\infty}(X)$  for the  $\mathbb{T}$ -equivariant  $\Omega$ -spectrum

$$\bar{\Sigma}_{\mathbb{T}}^{\infty}(X)(n) = \operatorname{hocolim}_V \Omega^V(X \wedge S^V \wedge S^n),$$

where the homotopy colimit is over the set of representations in  $\mathcal{V}$ . Similarly, let  $\mathbb{E}\bar{\Sigma}_{\mathbb{T}}^{\infty}(X)$  be the symmetric spectrum with  $\mathbb{T}$ -action defined by

$$\mathbb{E}\bar{\Sigma}_{\mathbb{T}}^{\infty}(X)(n) = \operatorname{hocolim}_V \Omega^V(\mathbb{E}^{\infty}(X \wedge S^V \wedge S^n)). \quad (4.4)$$

It follows from [27] that the natural chain of maps

$$\mathbb{E}\Sigma_{\mathbb{T}}^{\infty}(X) \rightarrow \mathbb{E}\bar{\Sigma}_{\mathbb{T}}^{\infty}(X) \leftarrow \bar{\Sigma}_{\mathbb{T}}^{\infty}(X)$$

restrict to equivalences on the associated  $C_r$  fixed-point spectra for all  $r$ . We shall use the notation  $\mathbb{E}\Sigma^{\infty}(X)$  for the underlying non-equivariant spectrum obtained by ignoring the  $\mathbb{T}$ -action.

#### 4.3. The combinatorial restriction maps

Given a based monoid  $\Pi$  we let  $B^{\text{cy}}(\Pi)$  be the based version of Waldhausen’s cyclic bar construction. This is the realization of the based cyclic set

$$B_{\bullet}^{\text{cy}}(\Pi): [k] \mapsto \Pi^{\wedge(k+1)}$$

with the usual Hochschild type structure maps, see e.g. [14, 3.1]. In this section we adapt the constructions in [27] to give *combinatorial restriction maps*

$$t_n: \mathbb{E}\Sigma_{\mathbb{T}}^{\infty}(B^{\text{cy}}(\Pi_{mn})) \rightarrow \mathbb{E}\Sigma_{\mathbb{T}}^{\infty}(B^{\text{cy}}(\Pi_m)). \quad (4.5)$$

These are equivariant maps of  $\mathbb{T}$ -spectra and provide models of the TH-restriction maps  $t_n$  in the sense of the following proposition.

**Proposition 4.6.** *There are diagrams of the form*

$$\begin{array}{ccc} \mathrm{TH}(\mathbb{S}(\Pi_{mn})) & \xrightarrow{\sim} & \mathbb{E}\Sigma_{\mathbb{T}}^{\infty}(B^{\mathrm{cy}}(\Pi_{mn})) \\ \downarrow t_n & & \downarrow t_n \\ \mathrm{TH}(\mathbb{S}(\Pi_m)) & \xrightarrow{\sim} & \mathbb{E}\Sigma_{\mathbb{T}}^{\infty}(B^{\mathrm{cy}}(\Pi_m)) \end{array}$$

in which the horizontal arrows represent chains of  $\mathbb{T}$ -equivariant spectrum maps, each of which (i) commutes with the induced restriction maps and (ii) restricts to equivalences on  $C_r$  fixed-point spectra for all  $r$ .

We first give an explicit description of the combinatorial restriction maps and shall return to the proof of the above proposition at the end of the section. Let  $M_n(\Pi_m)$  be the based monoid of  $n \times n$  matrices with entries in  $\Pi_m$  and at most one non-base point entry in each column. Letting  $\Pi_{mn}$  act on itself by left multiplication, we get an injective map of based monoids

$$\Phi: \Pi_{mn} \rightarrow M_n(\Pi_m).$$

Explicitly, the generator  $x$  is mapped to the matrix

$$\Phi(x) = \begin{pmatrix} 0 & x \\ I_{n-1} & 0 \end{pmatrix}.$$

Given an integer  $d \geq 0$ , write  $d$  in the form  $d = in + j$  for  $0 \leq i$  and  $0 \leq j \leq n - 1$ , and let

$$\varepsilon_d(s) = \begin{cases} i, & \text{for } 1 \leq s \leq n - j \\ i + 1, & \text{for } n - j + 1 \leq s \leq n. \end{cases}$$

Let  $\tau_n \in \Sigma_n$  be the  $n$ -cycle  $\tau_n = (1, 2, \dots, n)$ . The entries of  $\Phi(x)^d$  can then be written in the form

$$\Phi(x)_{hk}^d = \begin{cases} x^{\varepsilon_d(k)} & \text{for } h = \tau_n^j(k), \\ 0 & \text{for } h \neq \tau_n^j(k). \end{cases}$$

The basic ingredient in the definition of the combinatorial restriction map in (4.5) is the cyclic map

$$t_n: B_{\bullet}^{\mathrm{cy}}(\Pi_{mn}) \xrightarrow{\Phi} B_{\bullet}^{\mathrm{cy}}(M_n(\Pi_m)) \xrightarrow{\mathrm{tr}} \mathbb{E}_{\bullet}^{\infty}(B_{\bullet}^{\mathrm{cy}}(\Pi_m)), \quad (4.7)$$

where  $\mathrm{tr}$  is the trace map introduced in [25,27], and the target denotes the cyclic diagonal of the bicyclic set  $\mathbb{E}_h^{\infty}(B_k^{\mathrm{cy}}(\Pi_m))$ . Explicitly, given a  $k$ -simplex  $\omega = (x^{d_0}, \dots, x^{d_k})$ , it follows from the definition of  $\mathrm{tr}$  that  $t_n(\omega)$  is represented by the element

$$t'_n(\omega) = \begin{cases} [(\alpha_0, \dots, \alpha_k); (\mathbf{y}_1, \dots, \mathbf{y}_n)] & \text{for } d_0 + \dots + d_k = 0 \bmod n, \\ [(\alpha_0, \dots, \alpha_k); (*, \dots, *)] & \text{for } d_0 + \dots + d_k \neq 0 \bmod n, \end{cases} \quad (4.8)$$

in  $E_k \Sigma_n \times B_k^{\mathrm{cy}}(\Pi_m)^n$ . Here  $*$  denotes the base point of  $B_k^{\mathrm{cy}}(\Pi_m)$ , the permutations  $\alpha_s$  in  $\Sigma_n$  are defined by

$$\alpha_s = \begin{cases} \tau^{d_{s+1} + \dots + d_k} & \text{for } 0 \leq s \leq k - 1, \\ 1_n & \text{for } s = k, \end{cases}$$

and

$$\mathbf{y}_s = (x^{\varepsilon_{d_0}(\alpha_0(s))}, x^{\varepsilon_{d_1}(\alpha_1(s))}, \dots, x^{\varepsilon_{d_k}(\alpha_k(s))}) \quad \text{for } 1 \leq s \leq n.$$

The following lemma will be needed later.

**Lemma 4.9.** *If  $d_0 + \dots + d_k = 0 \bmod n$ , then*

$$\varepsilon_{d_0}(\alpha_0(s)) + \dots + \varepsilon_{d_k}(\alpha_k(s)) = (d_0 + \dots + d_k)/n$$

for  $1 \leq s \leq n$ .

**Proof.** Writing  $d_t = i_t n + j_t$  as in the definition of  $\varepsilon_{d_t}$ , we have by assumption that  $\sum_t j_t = rn$  for some  $r \geq 0$ . It is clear that there are exactly  $r$  of the indices  $t$  that satisfy

$$j_{t+1} + \dots + j_k + s \leq r_t n \quad \text{and} \quad j_t + \dots + j_k + s > r_t n$$

for some natural number  $r_t \geq 1$ . The result then follows from the definition of  $\varepsilon_{d_t}$ .  $\square$

We shall now use the structure maps of the cyclic Barratt–Eccles operad  $\{E_\bullet \Sigma_n : n \geq 0\}$  to extend the cyclic map in (4.7) to a map of cyclic  $\mathbb{T}$ -spectra

$$t_n : \mathbb{E}_\bullet \Sigma_{\mathbb{T}}^\infty(B_\bullet^{\text{cy}}(\Pi_{mn})) \rightarrow \mathbb{E}_\bullet \Sigma_{\mathbb{T}}^\infty(B_\bullet^{\text{cy}}(\Pi_m)).$$

We then define the combinatorial restriction map in (4.5) to be the topological realization. By definition [21, 15.1], the operad structure maps

$$\mu : E_\bullet \Sigma_n \times E_\bullet \Sigma_{j_1} \times \dots \times E_\bullet \Sigma_{j_n} \rightarrow E_\bullet \Sigma_{j_1 + \dots + j_n},$$

are given by

$$(\alpha, \beta_1, \dots, \beta_n) \mapsto \alpha\{j_1, \dots, j_n\} \circ (\beta_1 \sqcup \dots \sqcup \beta_n)$$

in each coordinate. Here  $\alpha\{j_1, \dots, j_n\}$  permutes the  $n$  blocks of letters as  $\alpha$  permutes the letters  $\{1, \dots, n\}$ . Applying these structure maps we get a natural transformation

$$\mu : \mathbb{E}_\bullet^\infty \mathbb{E}_\bullet^\infty(X) \rightarrow \mathbb{E}_\bullet^\infty(X)$$

and we may thus view  $\mathbb{E}_\bullet^\infty$  as a monad on the category of based cyclic spaces. In particular, this gives a canonical procedure for extending a map of based cyclic spaces  $X_\bullet \rightarrow \mathbb{E}_\bullet^\infty(Y_\bullet)$  to a map of cyclic  $\mathbb{T}$ -spectra

$$\mathbb{E}_\bullet \Sigma_{\mathbb{T}}^\infty(X_\bullet) \rightarrow \mathbb{E}_\bullet \Sigma_{\mathbb{T}}^\infty(Y_\bullet).$$

However, in the definition of the TH-restriction maps  $t_n$  we have used the reverse lexicographical ordering (2.2) and extending (4.7) in this fashion will not make the diagram in Proposition 4.6 commutative. In order to fix this, define the permutation  $B_{r,n}$  in  $\Sigma_{rn}$  to be the composite

$$B_{r,n} : \{1, \dots, rn\} \xrightarrow{l^{-1}} \{1, \dots, r\} \times \{1, \dots, n\} \xrightarrow{l^r} \{1, \dots, rn\},$$



where  $l$  and  $l'$  denote, respectively, the lexicographical and the reverse lexicographical orderings. For each representation  $V$  we then consider the composite map

$$\begin{aligned} E_k \Sigma_r \times (B_k^{\text{cy}}(\Pi_{mn}) \times S^V)^r &\xrightarrow{t'_n} E_k \Sigma_r \times (E_k \Sigma_n \times B_k^{\text{cy}}(\Pi_m)^n \times S^V)^r \\ &\longrightarrow E_k \Sigma_r \times (E_k \Sigma_n \times (B_k^{\text{cy}}(\Pi_m) \times S^V)^n)^r \\ &\longrightarrow E_k \Sigma_r \times (E_k \Sigma_n)^r \times (B_k^{\text{cy}}(\Pi_m) \times S^V)^{rn} \\ &\xrightarrow{\mu} E_k \Sigma_{rn} \times (B_k^{\text{cy}}(\Pi_m) \times S^V)^{rn} \\ &\xrightarrow{B_{r,n}} E_k \Sigma_{rn} \times (B_k^{\text{cy}}(\Pi_m) \times S^V)^{rn} \\ &\longrightarrow \mathbb{E}_k^\infty(B_k^{\text{cy}}(\Pi_m) \wedge S^V). \end{aligned}$$

The first map is given by the action of  $t'_n$  on the  $B_k^{\text{cy}}(\Pi_{mn})$  factors, the second map is induced by the diagonal inclusion of  $S^V$  in its  $n$ -fold product, the third map permutes the components in the obvious way, the fourth map is induced by the operad structure map, the fifth map is given by the left action of  $B_{r,n}$  on  $E_k \Sigma_{rn}$ , and the last map is the natural projection. We notice that the left action by  $B_{r,n}$  amounts to a change of basis in the sense of the equality

$$[B_{r,n} \mathbf{e}, \mathbf{x}] = [B_{n,r} \mathbf{e} B_{n,r}^{-1}, B_{n,r} \mathbf{x}].$$

The composite map is compatible with the defining relations on the domain for each  $k$  and induces a cyclic map

$$t_n(V): \mathbb{E}_\bullet^\infty(B_\bullet^{\text{cy}}(\Pi_{mn}) \wedge S^V) \rightarrow \mathbb{E}_\bullet^\infty(B_\bullet^{\text{cy}}(\Pi_m) \wedge S^V)$$

whose topological realization is the  $V$ th map of the combinatorial restriction map. This definition makes the action of the combinatorial restriction maps strictly transitive in the sense that  $t_m t_n = t_{mn}$  and we have the following analogue of Proposition 3.3.

**Lemma 4.10.** *The projection and transfer maps define an  $\mathbb{I}$ -diagram*

$$p_n, t_n: \mathbb{E}\Sigma_{\mathbb{T}}^\infty(B^{\text{cy}}(\Pi_{mn})) \rightarrow \mathbb{E}\Sigma_{\mathbb{T}}^\infty(B^{\text{cy}}(\Pi_m))$$

of  $\mathbb{T}$ -spectra.

**Proof of Proposition 4.6.** We proceed as in [27] and construct a commutative diagram of the form

$$\begin{array}{ccccc} \text{TH}(\mathbb{S}(\Pi_{mn})) & \xrightarrow{\sim} & \text{TH}\mathbb{E}^\infty(B^{\text{cy}}(\Pi_{mn})) & \xleftarrow{\sim} & \mathbb{E}\Sigma_{\mathbb{T}}^\infty(B^{\text{cy}}(\Pi_{mn})) \\ \downarrow t_n & & \downarrow t_n & & \downarrow t_n \\ \text{TH}(\mathbb{S}(\Pi_m)) & \xrightarrow{\sim} & \text{TH}\mathbb{E}^\infty(B^{\text{cy}}(\Pi_m)) & \xrightarrow{\sim} & \mathbb{E}\Sigma_{\mathbb{T}}^\infty(B^{\text{cy}}(\Pi_m)). \end{array}$$

The horizontal maps are defined as in [27]. Using the same kind of “change of basis” as in the definition of the combinatorial restriction map, we then define the vertical map  $t_n$  in the middle so as to make the diagram commutative.  $\square$

Let  $\tilde{\mathbb{E}}\Sigma_{\mathbb{T}}^{\infty}(B^{\text{cy}}(\Pi_n))$  be the homotopy fiber of the projection

$$\mathbb{E}\Sigma_{\mathbb{T}}^{\infty}(B^{\text{cy}}(\Pi_n)) \rightarrow \mathbb{E}\Sigma_{\mathbb{T}}^{\infty}(S^0).$$

**Corollary 4.11.** *The equivalences in Proposition 4.6 give rise to a chain of equivalence of  $\mathbb{I}$ -diagrams*

$$(\tilde{\mathbb{E}}\Sigma_{\mathbb{T}}^{\infty}(B^{\text{cy}}(\Pi_n)) \wedge_{\mathcal{V}} \text{TH}(A))^{C_r} \simeq (\tilde{\text{TH}}(\mathbb{S}(\Pi_n)) \wedge_{\mathcal{V}} \text{TH}(A))^{C_r}$$

for each  $r$ .

## 5. Topological Hochschild homology of truncated spherical polynomial rings

As in the linear case we let

$$\text{TF}(\mathbb{S}(\Pi_n)) = \text{holim}_{F_r} \text{TH}(\mathbb{S}(\Pi_n))^{C_r}.$$

In this section we shall analyse the  $\mathbb{I}$ -diagram  $n \mapsto \tilde{\text{TF}}(\mathbb{S}(\Pi_n))$ . This will serve as a model of the analysis of the analogous  $\mathbb{I}$ -diagram  $n \mapsto \tilde{\text{TF}}(A_n)$ . Using the pairings

$$\tilde{\text{TF}}(\mathbb{S}(\Pi_n)) \wedge \text{TF}(\mathbb{S}) \rightarrow \tilde{\text{TF}}(\mathbb{S}(\Pi_n)), \quad (5.1)$$

a class  $v$  in  $\pi_1 \text{holim}_{\mathbb{I}} \tilde{\text{TF}}(\mathbb{S}(\Pi_n))$  gives rise to a stable map

$$v: \Sigma \text{TF}(\mathbb{S}) \rightarrow \text{holim}_{\mathbb{I}} \tilde{\text{TF}}(\mathbb{S}(\Pi_n)).$$

The following is the main result of this section.

**Proposition 5.2.** *There exists a class  $v_{\infty}$  such that the induced map of completions*

$$v_{\infty}: \Sigma \text{TF}(\mathbb{S})^{\wedge} \xrightarrow{\sim} \text{holim}_{\mathbb{I}} \tilde{\text{TF}}(\mathbb{S}(\Pi_n))^{\wedge}$$

is an equivalence.

In order to prove this we shall use that the equivalences in Proposition 4.6 induce a chain of equivalences

$$\text{holim}_{\mathbb{I}} \tilde{\text{TF}}(\mathbb{S}(\Pi_n)) \simeq \text{holim}_{\mathbb{I}} \tilde{\mathbb{E}}\Sigma_F^{\infty}(B^{\text{cy}}(\Pi_n)), \quad (5.3)$$

where we use the notation

$$\tilde{\mathbb{E}}\Sigma_F^{\infty}(B^{\text{cy}}(\Pi_n)) = \text{holim}_{F_r} \tilde{\mathbb{E}}\Sigma_{\mathbb{T}}^{\infty}(B^{\text{cy}}(\Pi_n))^{C_r}.$$

We begin by analyzing the right hand side. Following [15, 7.2] we define the total degree of a  $k$ -simplex  $(x^{d_0}, \dots, x^{d_k})$  in  $B_k^{\text{cy}}(\Pi_n)$  to be the sum  $d_0 + \dots + d_k$ , where we say that 0 has all degrees. Letting  $B_{\bullet}^{\text{cy}}(\Pi_n, d)$  be the cyclic subset containing the simplices of total degree  $d$ , we get an isomorphism of cyclic sets

$$B_{\bullet}^{\text{cy}}(\Pi_n) \cong \bigvee_{d=0}^{\infty} B_{\bullet}^{\text{cy}}(\Pi_n, d).$$

Since the  $k$ -skeleton of  $B^{\text{cy}}(\Pi_n, d)$  is a point for  $k < d/(n-1) - 1$ , the connectivity of these spaces tends to infinity with  $d$ . Similarly, one checks that the connectivity of the fixed-point spaces  $B^{\text{cy}}(\Pi_n, d)^{C_r}$  tends to infinity with  $d$  by considering their edgewise subdivisions [7,1]. Applying  $\mathbb{E}\Sigma_{\mathbb{T}}^{\infty}$  and taking fixed points we get a canonical stable equivalence

$$\tilde{\mathbb{E}}\Sigma_{\mathbb{T}}^{\infty}(B^{\text{cy}}(\Pi_n))^{C_r} \xrightarrow{\sim} \prod_{d=1}^{\infty} \mathbb{E}\Sigma_{\mathbb{T}}^{\infty}(B^{\text{cy}}(\Pi_n, d))^{C_r} \quad (5.4)$$

for each  $n$  and  $r$ . We next consider the effect on  $B^{\text{cy}}(\Pi_n, d)$  of letting  $n$  tend to infinity. For fixed  $d$ ,

$$B^{\text{cy}}(\Pi_{d+1}, d) = B^{\text{cy}}(\Pi_{d+2}, d) = \cdots = B^{\text{cy}}(\Pi_{\infty}, d),$$

where  $\Pi_{\infty}$  denotes the based monoid of monomials (including 0) in  $\mathbb{Z}[x]$ . We write  $Z_{\bullet}(d)$  for the cyclic set  $B_{\bullet}^{\text{cy}}(\Pi_{\infty}, d)$  and  $Z(d)$  for its realization. Notice that  $(1, x)$  is the unique non-degenerate 1-simplex in  $Z_{\bullet}(1)$  and that the latter may be identified with the union  $S^1_{\bullet+}$  of the standard simplicial circle and a disjoint base point. The equivariant homotopy type of  $Z(d)$  has been determined by Hesselholt in general. Let  $S^1(d)$  denote the unit circle in the  $\mathbb{T}$ -representation  $\mathbb{C}(d)$  considered in Section 4.

**Lemma 5.5** (Hesselholt [14, 2.2.3]). *The  $\mathbb{T}$ -space  $Z(d)$  contains  $S^1(d)_+$  as a strong equivariant deformation retract.*

We now return to the homotopy limits in (5.3).

**Lemma 5.6.** *There is a chain of equivalences*

$$\text{holim}_{\mathbb{I}} \tilde{\mathbb{E}}\Sigma_F^{\infty}(B^{\text{cy}}(\Pi_n)) \simeq \text{holim}_{t_n \circ F_n} \mathbb{E}\Sigma_{\mathbb{T}}^{\infty}(Z(n))^{C_n},$$

where the last homotopy limit is over the composite maps

$$\mathbb{E}\Sigma_{\mathbb{T}}^{\infty}(Z(mn))^{C_{mn}} \xrightarrow{F_n} \mathbb{E}\Sigma_{\mathbb{T}}^{\infty}(Z(mn))^{C_m} \xrightarrow{t_n} \mathbb{E}\Sigma_{\mathbb{T}}^{\infty}(Z(m))^{C_m}.$$

**Proof.** We consider the  $\tilde{N} \times \mathbb{I}$ -diagram

$$(r, n) \mapsto \tilde{\mathbb{E}}\Sigma_{\mathbb{T}}^{\infty}(B^{\text{cy}}(\Pi_n))^{C_r}$$

and observe that

$$\text{holim}_{\mathbb{I}} \text{holim}_{F_r} \tilde{\mathbb{E}}\Sigma_{\mathbb{T}}^{\infty}(B^{\text{cy}}(\Pi_n))^{C_r} = \text{holim}_{F_r} \text{holim}_{\mathbb{I}} \tilde{\mathbb{E}}\Sigma_{\mathbb{T}}^{\infty}(B^{\text{cy}}(\Pi_n))^{C_r}.$$

The equivalences in (5.4) specify a level-wise equivalence of  $\tilde{N} \times \mathbb{I}$ -diagrams, and using (2.6) together with the fact that homotopy limits commute with infinite products we get that

$$\text{holim}_{\mathbb{I}} \tilde{\mathbb{E}}\Sigma_{\mathbb{T}}^{\infty}(B^{\text{cy}}(\Pi_n))^{C_r} \simeq \left( \prod_{d=1}^{\infty} \mathbb{E}\Sigma_{\mathbb{T}}^{\infty}(Z(d))^{C_r} \right)^{h\{t_n\}}.$$

Using Lemma 4.9 we see that (i)  $t_n$  acts trivially on the factors  $\mathbb{E}\Sigma_{\mathbb{T}}^{\infty}(Z(d))^{C_r}$  for which  $n$  does not divide  $d$  and that (ii) the action of  $t_n$  on the components for which  $n$  divides  $d$  factors through the  $(d/n)$ th

components,

$$t_n : \mathbb{E}\Sigma_{\mathbb{T}}^{\infty}(Z(d))^{C_r} \rightarrow \mathbb{E}\Sigma_{\mathbb{T}}^{\infty}(Z(d/n))^{C_r}.$$

It follows that the action of the monoid  $N$  on the infinite product via the restriction maps  $t_n$  results from a diagram over the associated translation category  $\tilde{N}$  as in Lemma 2.4 and consequently

$$\operatorname{holim}_{\tilde{N}} \left( \prod_{d=1}^{\infty} \mathbb{E}\Sigma_{\mathbb{T}}^{\infty}(Z(d))^{C_r} \right)^{h\{t_n\}} \simeq \operatorname{holim}_{\tilde{N} \times \tilde{N}} \mathbb{E}\Sigma_{\mathbb{T}}^{\infty}(Z(d))^{C_r}.$$

The result now follows from the fact that the diagonal embedding of  $\tilde{N}$  in the product is left cofinal in the sense of [4, XI, 9.3].  $\square$

We next specify a canonical map

$$\bar{v}_n : \Sigma^{\infty}(Z(1)) \rightarrow \mathbb{E}\Sigma_{\mathbb{T}}^{\infty}(Z(n))^{C_n} \quad (5.7)$$

for each  $n$ . The definition uses the edgewise subdivision functor  $\operatorname{sd}_n$  from [7]. We recall that  $\operatorname{sd}_n$  is an endofunctor on the category of simplicial spaces and that the topological realizations of  $X_{\bullet}$  and  $\operatorname{sd}_n X_{\bullet}$  are related by a natural homeomorphism  $D_n : |\operatorname{sd}_n X_{\bullet}| \rightarrow |X_{\bullet}|$ . Furthermore, if  $X_{\bullet}$  has the structure of a cyclic set, then  $\operatorname{sd}_n X_{\bullet}$  has a simplicial  $C_n$ -action and  $D_n$  is  $C_n$ -equivariant. Given a based monoid  $\Pi$ , the edgewise subdivision of  $B_{\bullet}^{\operatorname{cy}}(\Pi)$  may be written in the form

$$\operatorname{sd}_n B_{\bullet}^{\operatorname{cy}}(\Pi) \cong B_{\bullet}^{\operatorname{cy}}(\tau\Pi^{\wedge n}, \Pi^{\wedge n}),$$

where  $\tau\Pi^{\wedge n}$  denotes the based set  $\Pi^{\wedge n}$  with a twisted two-sided  $\Pi^{\wedge n}$ -action, see [7, 2]. From this description it is clear that the diagonal inclusion gives a simplicial isomorphism

$$B_{\bullet}^{\operatorname{cy}}(\Pi) \xrightarrow{\sim} \operatorname{sd}_n B_{\bullet}^{\operatorname{cy}}(\Pi)^{C_n}.$$

Consider now the composition

$$Z_{\bullet}(1) \xrightarrow{\sim} \operatorname{sd}_n Z_{\bullet}(n)^{C_n} \rightarrow \operatorname{sd}_n \mathbb{E}_{\bullet}^{\infty}(Z_{\bullet}(n))^{C_n},$$

in which the first map is the diagonal inclusion and the second map is the edgewise subdivision of the natural inclusion  $Z_{\bullet}(n) \rightarrow \mathbb{E}_{\bullet}^{\infty}(Z_{\bullet}(n))$ . After topological realization and composing with the homeomorphism  $D_n$ , we get a map of spaces

$$Z(1) \rightarrow \operatorname{sd}_n \mathbb{E}^{\infty}(Z(n))^{C_n} \xrightarrow{D_n^{C_n}} \mathbb{E}^{\infty}(Z(n))^{C_n}.$$

The map  $\bar{v}_n$  in (5.7) is the extension to a map of spectra. In order to formulate the next lemma, we specify a stable self-map  $\psi_n$  of  $\Sigma^{\infty}(Z(1))$  as follows. We identify  $Z(1)$  and  $S_+^1$  as above. The cofibration sequence  $S^0 \rightarrow S_+^1 \rightarrow S^1$  is stably split by the projection  $S_+^1 \rightarrow S^0$  and there results a canonical decomposition

$$\Sigma^{\infty}(S_+^1) \simeq \Sigma^{\infty}(S^1) \vee \Sigma^{\infty}(S^0) \quad (5.8)$$

in the stable homotopy category. Under this identification we specify that  $\psi_n$  acts as the matrix

$$\begin{pmatrix} 1 & 0 \\ (n-1)\eta & n \end{pmatrix},$$

where  $\eta \in \pi_1^s(S^0)$  denotes the stable Hopf map.

**Lemma 5.9.** *The diagram*

$$\begin{array}{ccc} \Sigma^\infty(Z(1)) & \xrightarrow{\psi_n} & \Sigma^\infty(Z(1)) \\ \downarrow \bar{v}_{mn} & & \downarrow \bar{v}_m \\ \mathbb{E}\Sigma_{\mathbb{T}}^\infty(Z(mn))^{C_{mn}} & \xrightarrow{t_n \circ F_n} & \mathbb{E}\Sigma_{\mathbb{T}}^\infty(Z(m))^{C_m} \end{array}$$

*is commutative in the stable homotopy category.*

In order to prove the above lemma we introduce an auxiliary map

$$\psi'_n: \Sigma^\infty(Z(1)) \rightarrow \mathbb{E}\Sigma^\infty(Z(1)).$$

Consider first the based simplicial map  $Z_\bullet(1) \rightarrow \mathbb{E}_\bullet^\infty(Z_\bullet(1))$  that sends the non-degenerate simplex  $(1, x)$  to the element represented by the 1-simplex

$$\left[ (\tau_n, 1); \underbrace{(x, 1), \dots, (x, 1)}_{n-1}, (1, x) \right] \in E_1 \Sigma_n \times Z_1(1)^n.$$

We then define  $\psi'_n$  to be the unique map of spectra that in degree zero is the realization of this map. Let  $i: \Sigma^\infty(Z(1)) \rightarrow \mathbb{E}\Sigma^\infty(Z(1))$  be the canonical equivalence.

**Lemma 5.10.** *The diagram*

$$\begin{array}{ccc} \Sigma^\infty(Z(1)) & \xrightarrow{\psi_n} & \Sigma^\infty(Z(1)) \\ \parallel & & \downarrow i \\ \Sigma^\infty(Z(1)) & \xrightarrow{\psi'_n} & \mathbb{E}\Sigma^\infty(Z(1)) \end{array}$$

*is commutative in the stable homotopy category.*

**Proof.** Notice first that the cofibration sequence  $S^0 \rightarrow S^1_+ \rightarrow S^1$  gives rise to a strictly commutative diagram of spectra

$$\begin{array}{ccccc} \Sigma^\infty(S^0) & \longrightarrow & \Sigma^\infty(Z(1)) & \longrightarrow & \Sigma^\infty(S^1) \\ \downarrow n \cdot i & & \downarrow \psi'_n & & \downarrow i \\ \mathbb{E}\Sigma^\infty(S^0) & \longrightarrow & \mathbb{E}\Sigma^\infty(Z(1)) & \longrightarrow & \mathbb{E}\Sigma^\infty(S^1) \end{array}$$

in which the left hand vertical map is the  $n$ -fold multiple of the natural equivalence  $i$ . It remains to identify the stable homotopy class

$$\Sigma^\infty(S^1) \rightarrow \Sigma^\infty(Z(1)) \xrightarrow{\psi'_n} \mathbb{E}\Sigma^\infty(Z(1)) \rightarrow \mathbb{E}\Sigma^\infty(S^0),$$

where the first map is induced by the stable splitting (5.8). Notice that  $\mathbb{E}^\infty(S^0) = \coprod_m B\Sigma_m$  and let  $b_m$  denote the base point of  $B\Sigma_m$ . The permutation  $\tau_n$  gives rise to a canonical map  $\bar{\tau}_n: S^1 \rightarrow B\Sigma_n$ , and the above stable homotopy class is represented by the composition

$$S^1 \xrightarrow{\bar{\tau}_n} B\Sigma_n \rightarrow \mathbb{E}^\infty(S^0) \rightarrow \Omega\mathbb{E}^\infty(S^1) \xrightarrow{-b_n} \Omega\mathbb{E}^\infty(S^1)$$

where the last map is multiplication by  $-b_n$  using the group-like H-space structure. It follows from [1, 5.2, 5.4] that the canonical map  $\mathbb{E}^\infty(S^0) \rightarrow \Omega\mathbb{E}^\infty(S^1)$  is a group completion in the sense that the homology of the target is obtained by inverting the class represented by  $b_1$  in the Pontryagin ring structure

$$H_*(\mathbb{E}^\infty(S^0))[b_1^{-1}] \xrightarrow{\sim} H_*(\Omega\mathbb{E}^\infty(S^1)).$$

Restricting to the base point components this isomorphism is induced by a homology equivalence  $B\Sigma_\infty \rightarrow \Omega_0\mathbb{E}^\infty(S^1)$  and the above homotopy class admits a factorization

$$S^1 \xrightarrow{\bar{\tau}_n} B\Sigma_n \rightarrow B\Sigma_\infty \rightarrow \Omega_0\mathbb{E}^\infty(S^1).$$

The result now follows since on fundamental groups the last map is given by the signature  $\Sigma_\infty \rightarrow \{\pm 1\}$ .  $\square$

We shall also need an extension of  $\bar{v}_n$  to a map of spectra

$$\bar{v}'_n: \mathbb{E}^{\Sigma^\infty}(Z(1)) \rightarrow \mathbb{E}^{\Sigma^\infty}_{\mathbb{T}}(Z(n))^{C_n}.$$

Notice first that the edgewise subdivision  $\text{sd}_n E_\bullet \Sigma_m$  is isomorphic to  $E_\bullet \Sigma_m^n$  with  $C_n$  acting by cyclic permutation of the coordinates. Using this, we see that

$$\text{sd}_n \mathbb{E}^\infty_\bullet(Z_\bullet(n)) = \coprod_{m=0}^{\infty} E_\bullet \Sigma_m^n \times B_\bullet^{\text{cy}}(\tau \Pi_\infty^{\wedge n}, \Pi_\infty^{\wedge n})^m / \sim,$$

where  $\sim$  is the equivalence relation inherited from  $\mathbb{E}^\infty_\bullet(Z_\bullet(n))$ . Using the diagonal inclusions

$$E_\bullet \Sigma_m \rightarrow E_\bullet \Sigma_m^n \quad \text{and} \quad B_\bullet^{\text{cy}}(\Pi_\infty) \rightarrow B_\bullet^{\text{cy}}(\tau \Pi_\infty^{\wedge n}, \Pi_\infty^{\wedge n}),$$

we get a map of simplicial spectra

$$\mathbb{E}_\bullet \Sigma^\infty(Z_\bullet(1)) \rightarrow \text{sd}_n \mathbb{E}_\bullet \Sigma^\infty(Z_\bullet(n))^{C_n}$$

and  $\bar{v}'_n$  is obtained from this by topological realization and applying the homeomorphism  $D_n$ .

**Proof of Lemma 5.9.** We claim that there is a homotopy commutative diagram

$$\begin{array}{ccccc}
 \Sigma^\infty(Z(1)) & \xrightarrow{\psi'_n} & \mathbb{E}\Sigma^\infty(Z(m)) & \xleftarrow{i} & \Sigma^\infty(Z(1)) \\
 \downarrow \bar{v}_{mn} & & \downarrow \bar{v}'_m & & \downarrow \bar{v}_m \\
 \mathbb{E}\Sigma^\infty(Z(mn))^{C_{mn}} & \xrightarrow{t_n \circ F_n} & \mathbb{E}\Sigma^\infty(Z(m))^{C_m} & \xlongequal{\quad} & \mathbb{E}\Sigma^\infty(Z(m))^{C_m}.
 \end{array}$$

Assuming this, the result follows from Lemma 5.10. By the definition of  $\bar{v}'_m$ , it is clear that the right hand square is commutative. For the left hand square it suffices to show that the underlying diagram of spaces in spectrum degree zero is homotopy commutative. In order to do this, we recall that given a simplicial set  $X_\bullet$ , the edgewise subdivision  $\text{sd}_n X_\bullet$  is related to  $X_\bullet$  via a simplicial map  $\tilde{D}_n$  defined by

$$\tilde{D}_{n,k} = d_0^{(n-1)(k+1)}: \text{sd}_n X_k \rightarrow X_k.$$

It follows from the proof of [7, 2.5] that the realization of  $\tilde{D}_n$  is homotopic to the (non-simplicial) homeomorphism  $D_n$ , and in particular that  $\tilde{D}_n$  is an equivalence. Consider the simplicial map

$$\begin{aligned}
 \text{sd}_{mn} \mathbb{E}_\bullet^\infty(Z_\bullet(mn))^{C_{mn}} &\rightarrow \text{sd}_{mn} \mathbb{E}_\bullet^\infty(Z_\bullet(mn))^{C_m} \cong \text{sd}_n \text{sd}_m \mathbb{E}_\bullet^\infty(Z_\bullet(mn))^{C_m} \\
 &\xrightarrow{\tilde{D}_n} \text{sd}_m \mathbb{E}_\bullet^\infty(Z_\bullet(mn))^{C_m} \xrightarrow{\text{sd}_m(t_n)} \text{sd}_m \mathbb{E}_\bullet^\infty(Z_\bullet(m))^{C_m},
 \end{aligned}$$

in which the first map is the inclusion. Using the naturality of  $\tilde{D}_n$  and the (non-equivariant) homotopy relating the latter to  $D_n$ , we see that the realization of this map is homotopic to the composition

$$\mathbb{E}^\infty(Z(mn))^{C_{mn}} \xrightarrow{F_n} \mathbb{E}^\infty(Z(mn))^{C_m} \xrightarrow{t_n} \mathbb{E}^\infty(Z(m))^{C_m}$$

upon identifying the domains and targets via  $D_{mn}$  and  $D_m$  respectively. It thus suffices to check the commutativity of the diagram of simplicial sets

$$\begin{array}{ccc}
 Z_\bullet(1) & \xrightarrow{\psi'_n} & \mathbb{E}_\bullet^\infty(Z_\bullet(1)) \\
 \downarrow \bar{v}_{mn} & & \downarrow \bar{v}'_m \\
 \text{sd}_{mn} \mathbb{E}_\bullet^\infty(Z_\bullet(mn))^{C_{mn}} & \longrightarrow & \text{sd}_m \mathbb{E}_\bullet^\infty(Z_\bullet(m))^{C_m},
 \end{array}$$

where  $\psi'_n$ ,  $\bar{v}_{mn}$  and  $\bar{v}_m$  denote the simplicial maps on which the corresponding spectrum maps are based. The lower horizontal map is the composite simplicial map considered above. Since this is a diagram of based simplicial sets and the only non-degenerate simplex of  $Z_\bullet(1)$  is the 1-simplex  $(1, x)$ , the result follows from the fact that the two compositions in the diagram take the same value on this element.  $\square$

We now define  $v_n$  to be the stable homotopy class

$$v_n: \Sigma^\infty(S^1) \xrightarrow{(1, \eta)} \Sigma^\infty(Z(1)) \xrightarrow{\bar{v}_n} \mathbb{E}\Sigma_{C_n}^\infty(Z(n))^{C_n},$$

where the first map is, respectively, the identity and multiplication by  $\eta$  on the two summands in the decomposition (5.8) of  $\Sigma^\infty(Z(1))$ .

**Proposition 5.11.** *The classes  $v_n$  are compatible and determine a stable homotopy class*

$$v_\infty: \Sigma^\infty(S^1) \rightarrow \operatorname{holim}_{t_n \circ F_n} \mathbb{E}\Sigma_{C_n}^\infty(Z(n))^{C_n}.$$

**Proof.** Since the groups  $\pi_2 \mathbb{E}\Sigma_{\mathbb{T}}^\infty(Z(n))^{C_n}$  are finite, the  $\lim^1$ -term in the Milnor exact sequence vanishes and we get an isomorphism

$$\pi_1 \operatorname{holim} \mathbb{E}\Sigma_{\mathbb{T}}^\infty(Z(n))^{C_n} \xrightarrow{\sim} \lim \pi_1 \mathbb{E}\Sigma_{\mathbb{T}}^\infty(Z(n))^{C_n}.$$

By the definition of  $\psi_n$  we have that

$$\psi_n \begin{pmatrix} 1 \\ \eta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ (n-1)\eta & n \end{pmatrix} \begin{pmatrix} 1 \\ \eta \end{pmatrix} = \begin{pmatrix} 1 \\ \eta \end{pmatrix}$$

and it thus follows from Lemma 5.9 that the classes  $v_n$  are compatible and specify an element in the inverse limit.  $\square$

We use the same notation for the stable homotopy class

$$v_\infty: \Sigma^\infty(S^1) \rightarrow \operatorname{holim}_{t_n \circ F_n} \mathbb{E}\Sigma^\infty(Z(n))^{C_n} \simeq \operatorname{holim}_{\mathbb{I}} \tilde{\mathbb{T}}\mathbb{F}(\mathbb{S}(\Pi_n))$$

obtained by composing with the equivalences in (5.3) and Lemma 5.6.

**Proof of Proposition 5.2.** Recall the spectra  $\mathbb{E}\tilde{\Sigma}_{\mathbb{T}}^\infty(B^{\text{cy}}(\Pi_n))$  introduced in (4.4) and consider the pairings

$$\tilde{\mathbb{E}}\Sigma_F^\infty(B^{\text{cy}}(\Pi_n)) \wedge \tilde{\Sigma}_F^\infty(S^0) \rightarrow \tilde{\mathbb{E}}\tilde{\Sigma}_F^\infty(B^{\text{cy}}(\Pi_n)),$$

where we use the notation  $\tilde{\Sigma}_F^\infty(S^0)$  for the homotopy inverse limit of the fixed-point spectra  $\tilde{\Sigma}_{\mathbb{T}}^\infty(S^0)^{C_r}$  under the inclusion maps. These pairings are compatible with those in (5.1). Using Lemma 5.6, the class  $v_\infty$  corresponds to a class in  $\pi_1 \operatorname{holim}_{\mathbb{I}} \tilde{\mathbb{E}}\Sigma_F^\infty(B^{\text{cy}}(\Pi_n))$  and it suffices to show that the induced stable map

$$\Sigma \tilde{\Sigma}_F^\infty(S^0) \rightarrow \operatorname{holim}_{\mathbb{I}} \tilde{\mathbb{E}}\tilde{\Sigma}_F^\infty(B^{\text{cy}}(\Pi_n))$$

induces an equivalence after completion. It follows from Lemma 5.6 that the target is equivalent to a homotopy limit of the spectra  $\mathbb{E}\tilde{\Sigma}_{\mathbb{T}}^\infty(Z(n))^{C_n}$  and this homotopy limit may be calculated by restricting to a cofinal subsequence of  $\tilde{N}$ . It thus suffices to show that the maps of fixed-point spectra

$$v_n: \Sigma \tilde{\Sigma}_{\mathbb{T}}^\infty(S^0)^{C_n} \rightarrow \mathbb{E}\tilde{\Sigma}_{\mathbb{T}}^\infty(Z(n))^{C_n}$$

induced by the above pairings give rise to an equivalence of homotopy limits after completion. By definition, these maps are obtained by composing the maps

$$\Sigma \tilde{\Sigma}_{\mathbb{T}}^\infty(S^0)^{C_n} \xrightarrow{(1, \eta)} \Sigma^\infty(S_+^1) \wedge \tilde{\Sigma}_{\mathbb{T}}^\infty(S^0)^{C_n}$$

and

$$\Sigma^\infty(S_+^1) \wedge \tilde{\Sigma}_{\mathbb{T}}^\infty(S^0)^{C_n} \xrightarrow{\tilde{v}_n} \mathbb{E}\Sigma_{\mathbb{T}}^\infty(Z(n))^{C_n} \wedge \tilde{\Sigma}_{\mathbb{T}}^\infty(S^0)^{C_n} \rightarrow \mathbb{E}\tilde{\Sigma}_{\mathbb{T}}^\infty(Z(n))^{C_n}.$$

The composition defining the second map is an equivalence for each  $n$  since the diagonal inclusion  $S_+^1 \xrightarrow{\sim} Z(n)^{C_n} \rightarrow Z(n)$  is a  $C_n$ -equivariant homotopy equivalence. For the first map, we view the target



as an inverse system of spectra with structure maps  $\psi_n \wedge F_n$ . We may identify the homotopy limit of this system with the middle term in the cofibration sequence

$$\operatorname{holim}_n \bar{\Sigma}_F^\infty(S^0) \rightarrow \operatorname{holim}_{\psi_n} \Sigma^\infty(S_+^1) \wedge \bar{\Sigma}_F^\infty(S^0) \rightarrow \Sigma \bar{\Sigma}_F^\infty(S^0),$$

where the first term is the homotopy limit of the  $\tilde{N}$ -diagram in which the morphism  $mn \rightarrow m$  acts as multiplication by  $m$ . The completion of this term is contractible by Lemma 2.7 and it follows that the completion of the second arrow is a homotopy inverse of the map in question. This concludes the proof.  $\square$

## 6. The proof of Proposition 3.12

Let  $\mathrm{TH}(A)$  be the model of topological Hochschild homology considered in Section 4 and notice that there is a canonical *linearization map*

$$L: \tilde{\mathrm{TH}}(\mathbb{S}(\Pi_n)) \rightarrow \tilde{\mathrm{TH}}(\mathbb{Z}(\Pi_n)).$$

We may view this as a map of  $\mathbb{I}$ -diagrams and passing to homotopy limits over the associated fixed-point spectra we get a map

$$L: \operatorname{holim}_{\mathbb{I}} \tilde{\mathrm{TF}}(\mathbb{S}(\Pi_n)) \rightarrow \operatorname{holim}_{\mathbb{I}} \tilde{\mathrm{TF}}(\mathbb{Z}(\Pi_n)). \quad (6.1)$$

Using the pairings

$$\tilde{\mathrm{TH}}(\mathbb{S}(\Pi_n)) \wedge \mathrm{TH}(A) \rightarrow \tilde{\mathrm{TH}}(\mathbb{Z}(\Pi_n)) \wedge \mathrm{TH}(A) \rightarrow \tilde{\mathrm{TH}}(A_n),$$

we multiply by the stable class  $v_\infty$  from Proposition 5.2 to get a stable map

$$v_\infty: \Sigma \mathrm{TH}(A) \rightarrow \operatorname{holim}_{\mathbb{I}} \tilde{\mathrm{TF}}(A_n).$$

The stable class  $u_\infty^{\mathrm{TC}}$  defined in Section 3 corresponds under the equivalence (2.3) to a class in  $\pi_1 \operatorname{holim}_{\mathbb{I}} \tilde{\mathrm{TC}}(\mathbb{Z}(\Pi_n))$  and we let  $u_\infty^F$  be the image under the projection

$$\operatorname{holim}_{\mathbb{I}} \tilde{\mathrm{TC}}(\mathbb{Z}(\Pi_n)) \rightarrow \operatorname{holim}_{\mathbb{I}} \tilde{\mathrm{TF}}(\mathbb{Z}(\Pi_n)).$$

The proof of Proposition 3.12 now easily follows from the following two lemmas.

**Lemma 6.2.** *Multiplication by  $v_\infty$  gives an equivalences*

$$v_\infty: \Sigma \mathrm{TF}(A)^\wedge \rightarrow \operatorname{holim}_{\mathbb{I}} \tilde{\mathrm{TF}}(A_n)^\wedge.$$

**Lemma 6.3.** *The linearization map  $L$  in (6.1) satisfies*

$$Lv_\infty = u_\infty^F \in \pi_1 \operatorname{holim}_{\mathbb{I}} \tilde{\mathrm{TF}}(\mathbb{Z}(\Pi_n)).$$

**Proof of Proposition 3.12.** In the notation from Section 3, we must prove that the completion of the stable map

$$u_\infty^{\mathrm{TC}}: \Sigma \mathrm{TC}(A) \rightarrow \operatorname{holim}_{\mathbb{I}} \tilde{\mathrm{TC}}(A_n)$$

is an equivalence. Using the TF-analogue of (2.3), it follows from Lemmas 6.2 and 6.3 that we have a stable equivalence

$$u_{\infty}^F: \Sigma \mathrm{TF}(A)^{\wedge} \rightarrow \operatorname{holim}_{\mathbb{I}} \tilde{\mathrm{TF}}(A_n)^{\wedge}.$$

Choosing a representative for  $u_{\infty}^F$  we may realize this as a map of spectra that strictly commutes with the action of the  $R_r$ -maps. This follows from the fact that, as defined in Section 2.2, the image of the trace map from  $\mathrm{K}(\mathbb{Z}(\Pi_n))$  is contained in the  $\mathbb{T}$ -fixed points of  $\mathrm{TH}(\mathbb{Z}(\Pi_n))$ . The conclusion then follows by taking homotopy fixed-points as in Section 2.3.  $\square$

It remains to prove Lemmas 6.2 and 6.3.

**Proof of Lemma 6.2.** We use the equivalences in Proposition 4.3 and Corollary 4.11 together with splitting (5.4) to get a chain of equivalences of  $\tilde{N} \times \mathbb{I}$ -diagrams

$$\tilde{\mathrm{TH}}(A_n)^{C_r} \simeq \prod_{d=1}^{\infty} (\mathbb{E}\Sigma_{\mathbb{T}}^{\infty}(B^{\mathrm{cy}}(\Pi_n, d)) \wedge_{\mathcal{V}} \mathrm{TH}(A))^{C_r}.$$

Passing to homotopy limits and proceeding as in the proof of Lemma 5.6, this gives a chain of equivalences

$$\operatorname{holim}_{\mathbb{I}} \tilde{\mathrm{TF}}(A_n) \simeq \operatorname{holim}_{t_n \circ F_n} (\mathbb{E}\Sigma_{\mathbb{T}}^{\infty}(Z(n)) \wedge_{\mathcal{V}} \mathrm{TH}(A))^{C_n},$$

where the homotopy limit is over the  $\tilde{N}$ -diagram specified by the composite maps

$$\begin{aligned} (\mathbb{E}\Sigma_{\mathbb{T}}^{\infty}(Z(mn)) \wedge_{\mathcal{V}} \mathrm{TH}(A))^{C_{mn}} &\xrightarrow{F_n} (\mathbb{E}\Sigma_{\mathbb{T}}^{\infty}(Z(mn)) \wedge_{\mathcal{V}} \mathrm{TH}(A))^{C_m} \\ &\xrightarrow{t_n \wedge \mathrm{id}} (\mathbb{E}\Sigma_{\mathbb{T}}^{\infty}(Z(m)) \wedge_{\mathcal{V}} \mathrm{TH}(A))^{C_m}. \end{aligned}$$

By an argument similar to that in the proof of Proposition 5.2 one proves that the map of homotopy limits induced by the composite maps

$$\Sigma \mathrm{TH}(A)^{C_n} \xrightarrow{v_n \wedge \mathrm{id}} \mathbb{E}\Sigma_{\mathbb{T}}^{\infty}(Z(n))^{C_n} \wedge \mathrm{TH}(A)^{C_n} \rightarrow (\mathbb{E}\Sigma_{\mathbb{T}}^{\infty}(Z(n)) \wedge_{\mathcal{V}} \mathrm{TH}(A))^{C_n}$$

becomes an equivalence when completed. This implies the statement of the lemma.  $\square$

In preparation for the proof of Lemma 6.3 we recall some facts about the Dennis trace map. Given a ring  $A$ , let  $\mathrm{HH}(A)$  denote the topological realization of the usual Hochschild simplicial abelian group  $[k] \mapsto A^{\otimes(k+1)}$ . As for any simplicial abelian group, the homotopy groups  $\pi_i \mathrm{HH}(A)$  may be identified with the homology groups of the associated chain complex. There is a canonical map  $\pi_i \mathrm{TH}(A) \rightarrow \mathrm{HH}_i(A)$ , which is an isomorphism for  $i = 0, 1$ . The Dennis trace map

$$\mathrm{K}_i(A) \rightarrow \mathrm{HH}_i(A)$$

may be defined by precomposing with the trace map from Section 2.2. We shall use that for  $i = 1$ , the composite map

$$GL_1(A) \rightarrow \mathrm{K}_1(A) \rightarrow \mathrm{HH}_1(A)$$

maps a unit  $a$  to the homology class represented by  $a^{-1} \otimes a$ . This follows from [10, 2.1.6]. Let now  $A = \mathbb{Z}(\Pi_n)$  and notice that the decomposition of  $B^{\text{cy}}(\Pi_n)$  in Section 5 induces a decomposition

$$\mathrm{HH}_i(\mathbb{Z}(\Pi_n)) = \bigoplus_{d=0}^{\infty} \mathrm{HH}_i(\mathbb{Z}(\Pi_n), d)$$

for all  $i$ . In the case  $i = 1$  one finds that

$$\mathrm{HH}_1(\mathbb{Z}(\Pi_n), d) = \begin{cases} \mathbb{Z}\{x^{d-1} \otimes x\} & \text{for } 1 \leq d \leq n-1, \\ \mathbb{Z}/n\{x^{n-1} \otimes x\} & \text{for } d = n, \\ 0 & \text{otherwise,} \end{cases}$$

see [16, 2.1.5]. The following lemma motivates our choice of the units  $u_n = 1 + x + \cdots + x^{n-1}$  as opposed to their multiplicative inverses  $1 - x$ .

**Lemma 6.4.** *The composite homomorphism*

$$GL_1(\mathbb{Z}(\Pi_n)) \rightarrow K_1(\mathbb{Z}(\Pi_n)) \rightarrow \mathrm{HH}_1(\mathbb{Z}(\Pi_n)) \rightarrow \mathrm{HH}_1(\mathbb{Z}(\Pi_n), d)$$

takes  $u_n$  to the homology class represented by  $x^{d-1} \otimes x$  for  $d = 1, \dots, n$ .

**Proof.** By the above remarks,  $u_n$  is mapped to the class in  $\mathrm{HH}_1(\mathbb{Z}(\Pi_{n+1}))$  represented by

$$u_n^{-1} \otimes u_n = 1 \otimes 1 + \sum_{d=1}^{n-1} (1 \otimes x^d - x \otimes x^{d-1}) - x \otimes x^{n-1}.$$

Letting  $\partial$  denote the differential in the Hochschild complex,

$$\partial(1 \otimes x \otimes x^{d-1}) = x \otimes x^{d-1} - 1 \otimes x^d + x^{d-1} \otimes x$$

and the result follows.  $\square$

**Proof of Lemma 6.3.** Proceeding as in the proofs of Proposition 5.2 and Lemma 6.2, we get an equivalence

$$\mathrm{holim}_{\mathbb{I}} \tilde{\mathrm{TF}}(\mathbb{Z}(\Pi_n)) \simeq \mathrm{holim}_{t_n \circ F_n} (\mathbb{E}\Sigma_{\mathbb{T}}^{\infty}(Z(n)) \wedge_{\mathcal{V}} \mathrm{TH}(\mathbb{Z}))^{C_n},$$

and using that the composition  $S_+^1 \rightarrow Z(n)^{C_n} \rightarrow Z(n)$  is a  $C_n$ -equivariant homotopy equivalence, a further equivalence

$$\mathrm{holim}_{t_n \circ F_n} (\mathbb{E}\Sigma_{\mathbb{T}}^{\infty}(Z(n)) \wedge_{\mathcal{V}} \mathrm{TH}(\mathbb{Z}))^{C_n} \simeq \mathrm{holim}_{\psi_n \wedge F_n} \Sigma^{\infty}(S_+^1) \wedge \mathrm{TH}(\mathbb{Z})^{C_n}$$

with  $\psi_n$  as in Section 5. We now recall that the homotopy groups of  $\mathrm{TH}(\mathbb{Z})^{C_n}$  are finite in positive degrees. For  $n = 1$  this is a consequence of Bökstedt's explicit calculation [6] and the general case follows from the fundamental cofibration sequences [15, 2.2]. The  $\lim^1$ -term in the Milnor exact sequence thus vanishes and there results an isomorphism

$$\pi_i \mathrm{holim}_{\psi_n \wedge F_n} \Sigma^{\infty}(S_+^1) \wedge \mathrm{TH}(\mathbb{Z})^{C_n} \xrightarrow{\sim} \lim_{\psi_n \wedge F_n} \pi_i(S_+^1 \wedge \mathrm{TH}(\mathbb{Z})^{C_n})$$

for each  $i \geq 1$ . Consider then the cofibration sequences

$$S^0 \wedge \mathrm{TH}(\mathbb{Z})^{C_n} \rightarrow S_+^1 \wedge \mathrm{TH}(\mathbb{Z})^{C_n} \rightarrow S^1 \wedge \mathrm{TH}(\mathbb{Z})^{C_n}.$$

Using once more the finiteness of the groups  $\pi_i \mathrm{TH}(\mathbb{Z})^{C_n}$ , we see that the projection  $S_+^1 \rightarrow S^1$  induces an isomorphism

$$\lim_{\psi_n \wedge F_n} \pi_i(S_+^1 \wedge \mathrm{TH}(\mathbb{Z})^{C_n}) \xrightarrow{\sim} \lim_{\mathrm{id} \wedge F_n} \pi_i(S^1 \wedge \mathrm{TH}(\mathbb{Z})^{C_n}) \xrightarrow{\sim} \lim_{F_n} \pi_{i-1} \mathrm{TH}(\mathbb{Z})^{C_n},$$

again for  $i \geq 1$ . Let now  $i = 1$  and identify  $\pi_0 \mathrm{TH}(\mathbb{Z})^{C_n}$  with the ring of truncated Witt vectors  $\mathbb{W}_{\langle n \rangle}(\mathbb{Z})$  as in [15, Addendum 3.3]. Here  $\langle n \rangle$  denotes the truncation set of natural numbers dividing  $n$ . Applying the above chain of equivalences, we claim that both classes  $Lv_\infty$  and  $u_\infty^F$  project to the multiplicative unit in  $\mathbb{W}_{\langle n \rangle}(\mathbb{Z})$  for all  $n$ . This clearly implies the statement of the lemma. We first consider  $Lv_\infty$ . By an argument analogous to that for  $\mathbb{Z}(\Pi_n)$ , we have an equivalence

$$\mathrm{holim}_{\mathbb{I}} \tilde{\mathrm{TF}}(\mathbb{S}(\Pi_n)) \simeq \mathrm{holim}_{t_n \circ F_n} (\mathbb{E}\Sigma_{\mathbb{T}}^\infty(Z(n)) \wedge_{\mathcal{V}} \mathrm{TH}(\mathbb{S}))^{C_n},$$

which in turn induces an isomorphism

$$\pi_1 \mathrm{holim}_{\mathbb{I}} \tilde{\mathrm{TF}}(\mathbb{S}(\Pi_n)) \simeq \lim_{F_n} \pi_0 \mathrm{TH}(\mathbb{S})^{C_n}.$$

There are commutative diagrams of the form

$$\begin{array}{ccccc} S_+^1 \wedge \mathrm{TH}(\mathbb{S})^{C_n} & \xrightarrow{\sim} & (\mathbb{E}\Sigma_{\mathbb{T}}^\infty(Z(n)) \wedge_{\mathcal{V}} \mathrm{TH}(\mathbb{S}))^{C_n} & \xrightarrow{\sim} & \mathbb{E}\Sigma_{\mathbb{T}}^\infty(Z(n))^{C_n} \\ \uparrow \mathrm{id} \wedge 1 & & \uparrow \bar{v}_n \wedge 1 & & \uparrow \bar{v}_n \\ S_+^1 \wedge \mathbb{S} & \xrightarrow{\sim} & \Sigma^\infty(S_+^1) \wedge \mathbb{S} & \xleftarrow{\sim} & \Sigma^\infty(S_+^1) \end{array}$$

in the stable homotopy category, hence the class  $v_\infty$  projects to the unit in  $\pi_0 \mathrm{TH}(\mathbb{S})^{C_n}$  for all  $n$ . Since the above isomorphisms are compatible with the linearization map, the result follows from the fact that the latter is a map of ring spectra and in particular unit preserving. We now turn to  $u_\infty^F$ . Since  $\mathbb{Z}$  is torsion free, the ghost map [15,3]

$$w: \pi_0 \mathrm{TH}(\mathbb{Z})^{C_n} \cong \mathbb{W}_{\langle n \rangle}(\mathbb{Z}) \rightarrow \mathbb{Z}^{\langle n \rangle}$$

is injective and it suffices to evaluate the image of each of the ghost coordinates. Given a divisor  $d$  in  $\langle n \rangle$ , it follows from [15, Addendum 3.3] that the  $d$ th ghost coordinate is given by

$$F_d R_{n/d}: \pi_0 \mathrm{TH}(\mathbb{Z})^{C_n} \rightarrow \pi_0 \mathrm{TH}(\mathbb{Z})^{C_d} \rightarrow \pi_0 \mathrm{TH}(\mathbb{Z}) \cong \mathbb{Z}.$$

Consider the commutative diagram in the stable homotopy category

$$\begin{array}{ccccc}
 S_+^1 \wedge \mathrm{TH}(\mathbb{Z})^{C_n} & \xrightarrow{\sim} & (B^{\mathrm{cy}}(\Pi_{n+1}, n) \wedge \mathrm{TH}(\mathbb{Z}))^{C_n} & \longleftarrow & \mathrm{TH}(\mathbb{Z}(\Pi_{n+1}))^{C_n} \\
 \downarrow S_+^1 \wedge R_{n/d} & & \downarrow R_{n/d} & & \downarrow R_{n/d} \\
 S_+^1 \wedge \mathrm{TH}(\mathbb{Z})^{C_d} & \xrightarrow{\sim} & (B^{\mathrm{cy}}(\Pi_{n+1}, d) \wedge \mathrm{TH}(\mathbb{Z}))^{C_d} & \longleftarrow & \mathrm{TH}(\mathbb{Z}(\Pi_{n+1}))^{C_d} \\
 \downarrow S_+^1 \wedge F_d & & \downarrow F_d & & \downarrow F_d \\
 S_+^1 \wedge \mathrm{TH}(\mathbb{Z}) & \xrightarrow{\sim} & B^{\mathrm{cy}}(\Pi_{n+1}, d) \wedge \mathrm{TH}(\mathbb{Z}) & \longleftarrow & \mathrm{TH}(\mathbb{Z}(\Pi_{n+1})),
 \end{array}$$

where the horizontal arrows on the left hand side are induced by the  $C_d$ -equivariant homotopy equivalences  $S_+^1 \rightarrow B^{\mathrm{cy}}(\Pi_{n+1}, d)$  considered above. (Recall that  $Z(d) = B^{\mathrm{cy}}(\Pi_{n+1}, d)$  for  $d \leq n$ ). Under the isomorphism

$$\pi_0 \mathrm{TH}(\mathbb{Z}) \simeq \pi_1(S_+^1 \wedge \mathrm{TH}(\mathbb{Z})) \simeq \mathrm{HH}_1(\mathbb{Z}(\Pi_{n+1}), 1),$$

the multiplicative unit corresponds to the class represented by  $1 \otimes x$ , and the lower horizontal equivalence in the above diagram maps this class to the one represented by  $x^{d-1} \otimes x$  in

$$\mathrm{HH}_1(\mathbb{Z}(\Pi_{n+1}), d) \simeq \pi_1(B^{\mathrm{cy}}(\Pi_{n+1}, d) \wedge \mathrm{TH}(\mathbb{Z})).$$

This follows by an argument using the simplicial equivalence  $\tilde{D}_d$  considered in the proof of Lemma 5.9. Hence it suffices to prove that the composite homomorphism

$$GL_1(\mathbb{Z}(\Pi_{n+1})) \rightarrow K_1(\mathbb{Z}(\Pi_{n+1})) \rightarrow \mathrm{HH}_1(\mathbb{Z}(\Pi_{n+1})) \rightarrow \mathrm{HH}_1(\mathbb{Z}(\Pi_{n+1}), d)$$

maps  $u_{n+1}$  to the homology class represented by  $x^{d-1} \otimes x$ , and this is exactly the statement of Lemma 6.4.  $\square$

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